An understanding of the attributes and relationships of geometric objects can be applied in diverse contexts—interpreting a schematic drawing, estimating the amount of wood needed to frame a sloping roof, rendering computer graphics, or designing a sewing pattern for the most efficient use of material.

Although there are many types of geometry, school mathematics is devoted primarily to plane Euclidean geometry, studied both synthetically (without coordinates) and analytically (with coordinates). Euclidean geometry is characterized most importantly by the Parallel Postulate that through a point not on a given line there is exactly one parallel line. (Spherical geometry, in contrast, has no parallel lines.)

During high school, students begin to formalize their geometry experiences from elementary and middle school, using more precise definitions and developing careful proofs. Later in college some students develop Euclidean and other geometries carefully from a small set of axioms.

The concepts of congruence, similarity, and symmetry can be understood from the perspective of geometric transformation. Fundamental are the rigid motions: translations, rotations, reflections, and combinations of these, all of which are here assumed to preserve distance and angles (and therefore shapes generally). Reflections and rotations each explain a particular type of symmetry, and the symmetries of an object offer insight into its attributes—as when the reflective symmetry of an isosceles triangle assures that its base angles are congruent.

In the approach taken here, two geometric figures are defined to be congruent if there is a sequence of rigid motions that carries one onto the other. This is the principle of superposition. For triangles, congruence means the equality of all corresponding pairs of sides and all corresponding pairs of angles. During the middle grades, through experiences drawing triangles from given conditions, students notice ways to specify enough measures in a triangle to ensure that all triangles drawn with those measures are congruent. Once these triangle congruence criteria (ASA, SAS, and SSS) are established using rigid motions, they can be used to prove theorems about triangles, quadrilaterals, and other geometric figures.

Similarity transformations (rigid motions followed by dilations) define similarity in the same way that rigid motions define congruence, thereby formalizing the similarity ideas of “same shape” and “scale factor” developed in the middle grades. These transformations lead to the criterion for triangle similarity that two pairs of corresponding angles are congruent.

The definitions of sine, cosine, and tangent for acute angles are founded on right triangles and similarity, and, with the Pythagorean Theorem, are fundamental in many real-world and theoretical situations. The Pythagorean Theorem is generalized to non-right triangles by the Law of Cosines. Together, the Laws of Sines and Cosines embody the triangle congruence criteria for the cases where three pieces of information suffice to completely solve a triangle. Furthermore, these laws yield two possible solutions in the ambiguous case, illustrating that Side-Side-Angle is not a congruence criterion.

Analytic geometry connects algebra and geometry, resulting in powerful methods of analysis and problem solving. Just as the number line associates numbers with locations in one dimension, a pair of perpendicular axes associates pairs of numbers with locations in two dimensions. This correspondence between numerical coordinates and geometric points allows methods from algebra to be applied to geometry and vice versa. The solution set of an equation becomes a geometric curve, making visualization a tool for doing and understanding algebra. Geometric shapes can be described by equations, making algebraic manipulation into a tool for geometric understanding, modeling, and proof. Geometric transformations of the graphs of equations correspond to algebraic changes in their equations.

Dynamic geometry environments provide students with experimental and modeling tools that allow them to investigate geometric phenomena in much the same way as computer algebra systems allow them to experiment with algebraic phenomena.

Connections to Equations. The correspondence between numerical coordinates and geometric points allows methods from algebra to be applied to geometry and vice versa. The solution set of an equation becomes a geometric curve, making visualization a tool for doing and understanding algebra. Geometric shapes can be described by equations, making algebraic manipulation into a tool for geometric understanding, modeling, and proof.
Geometry Standards Overview

Note: The standards identified with a (+) contain additional mathematics that students should learn in order to take advanced courses such as calculus, advanced statistics or discrete mathematics that go beyond the mathematics that all students should study in order to be college- and career-ready. Explanations and examples of these standards are not included in this document.

Modeling Standards: Specific modeling standards appear throughout the high school standards indicated by a star symbol (★).

Congruence (G-CO)
- Experiment with transformations in the plane.
  - G.CO.1  G.CO.2  G.CO.3  G.CO.4  G.CO.5
- Understand congruence in terms of rigid motions.
  - G.CO.6  G.CO.7  G.CO.8
- Prove geometric theorems.
  - G.CO.9  G.CO.10  G.CO.11
- Make geometric constructions.
  - G.CO.12  G.CO.13

Similarity, Right Triangles, and Trigonometry (G-SRT)
- Understand similarity in terms of similarity transformations.
  - G.SRT.1  G.SRT.2  G.SRT.3
- Prove theorems involving similarity.
  - G.SRT.4  G.SRT.5
- Define trigonometric ratios and solve problems involving right triangles.
  - G.SRT.6  G.SRT.7  G.SRT.8 (★)
- Apply trigonometry to general triangles.
  - G.SRT.9 (+)  G.SRT.10 (+)  G.SRT.11 (+)

Circles (G-C)
- Understand and apply theorems about circles.
  - G.C.1  G.C.2  G.C.3  G.C.4 (+)
- Find arc lengths and areas of sectors of circles.
  - G.C.5

Expressing Geometric Properties with Equations (G-GPE)
- Translate between the geometric description and the equation for a conic section.
  - G.GPE.1  G.GPE.2  G.GPE.3 (+)
- Use coordinates to prove simple geometric theorems algebraically.
  - G.GPE.4  G.GPE.5  G.GPE.6  G.GPE.7 (★)

Geometric Measurement and Dimensions (G-GMD)
- Explain volume formulas and use them to solve problems.
  - G.GMD.1  G.GMD.2 (+)  G.GMD.3 (★)
- Visualize relationships between two-dimensional and three-dimensional objects.
  - G.GMD.4

Modeling with Geometry (G-MG) (★)
- Apply geometric concepts in modeling situations.
  - G.MG.1 (★)  G.MG.2 (★)  G.MG.3 (★)
### Geometry: Congruence (G-CO)

**Cluster:** *Experiment with transformations in the plane.*

**Standard: G.CO.1** Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.

**Suggested Standards for Mathematical Practice (MP):**

MP.6 Attend to precision.

**Connections: G.CO.1-5**

Rotations, reflections and translations are developed experimentally in the Grade 8, and this experience should be built upon in high school, giving greater attention to precise definitions and formal reasoning.

Transformations can be studied in terms of functions, where the inputs and outputs are points in the plane, rather than numbers.

Rotations are studied again in the cluster about circles.

**Explanations and Examples: G.CO.1**

Understand and use definitions of angles, circles, perpendicular lines, parallel lines, and line segments based on the undefined term of a point, a line, the distance along a line, and the length of an arc.

Define angles, circles, perpendicular lines, rays, and line segments precisely using the undefined terms and “if-then” and “if-only-if” statements.

**Examples:**
- Have students write their own understanding of a given term.
- Give students formal and informal definitions of each term and compare them.
- Develop precise definitions through use of examples and non-examples.
- Discuss the importance of having precise definitions.

**Instructional Strategies: G.CO.1-5**

Review vocabulary associated with transformations (e.g. point, line, segment, angle, circle, polygon, parallelogram, perpendicular, rotation reflection, translation).

Provide both individual and small-group activities, allowing adequate time for students to explore and verify conjectures about transformations and develop precise definitions of rotations, reflections and translations.

Provide real-world examples of rigid motions (e.g. Ferris wheels for rotation; mirrors for reflection; moving vehicles for translation).

Use graph paper, transparencies, tracing paper or dynamic geometry software to obtain images of a given figure under specified transformations.

*Continued on next page*
### Instructional Strategies: G.CO.1-5

Provide students with a pre-image and a final, transformed image, and ask them to describe the steps required to generate the final image. Show examples with more than one answer (e.g., a reflection might result in the same image as a translation).

Work backwards to determine a sequence of transformations that will carry (map) one figure onto another of the same size and shape.

Focus attention on the attributes (e.g. distances or angle measures) of a geometric figure that remain constant under various transformations.

Make the transition from transformations as physical motions to functions that take points in the plane as inputs and give other points as outputs. The correspondence between the initial and final points determines the transformation.

Analyze various figures (e.g. regular polygons, folk art designs or product logos) to determine which rotations and reflections carry (map) the figure onto itself. These transformations are the “symmetries” of the figure.

Emphasize the importance of understanding a transformation as the correspondence between initial and final points, rather than the physical motion.

Use a variety of means to represent rigid motions, including physical manipulatives, coordinate methods, and dynamic geometry software.

### Common Misconceptions: G.CO.1-5

The terms “mapping” and “under” are used in special ways when studying transformations. A translation is a type of transformation that moves all the points in the object in a straight line in the same direction.

Students should know that not every transformation is a translation.

Students sometimes confuse the terms “transformation” and “translation.”
**Geometry: Congruence (G-CO)**

**Cluster:** *Experiment with transformations in the plane.*

**Standard: G.CO.2** Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).

**Suggested Standards for Mathematical Practice (MP):**

MP.5  Use appropriate tools strategically.

MP.6  Attend to precision.

**Connections:**

Rotations, reflections and translations are developed experimentally in the Grade 8, and this experience should be built upon in high school, giving greater attention to precise definitions and formal reasoning.

Transformations can be studied in terms of functions, where the inputs and outputs are points in the plane, rather than numbers.

Rotations are studied again in the cluster about circles.

**Explanations and Examples: G.CO.2**

In middle school students have worked with translations, reflections, and rotations and informally with dilations. Point out the basis of rigid motions in geometric concepts, e.g., translations move points a specified distance along a line parallel to a specified line; rotations move objects along a circular arc with a specified center through a specified angle.

Use various technologies such as transparencies, geometry software, interactive whiteboards, and digital visual presenters to represent and compare rigid and size transformations of figures in a coordinate plane. Comparing transformations that preserve distance and angle to those that do not.

Describe and compare function transformations on a set of points as inputs to produce another set of points as outputs, to include translations and horizontal and vertical stretching.

Students may use geometry software and/or manipulatives to model and compare transformations.

**Examples:**

- Draw transformations of reflections, rotations, translations, and combinations of these using graph paper, transparencies and/or geometry software.
- Determine the coordinates for the image (output) of a figure when a transformation rule is applied to the preimage (input).
- Distinguish between transformations that are rigid (preserve distance and angle measure—reflections, rotations, translations, or combinations of these) and those that are not (dilations or rigid motions followed by dilations).

*Continued on next page*
Explanations and Examples: G.CO.2

- The figure below is reflected across the y-axis and then shifted up by 4 units. Draw the transformed figure and label the new coordinates.

What function can be used to describe these transformations in the coordinate plane?

Solution: 
\((-1x, y + 4)\)

Instructional Strategies: G.CO.1-5

Provide both individual and small-group activities, allowing adequate time for students to explore and verify conjectures about transformations and develop precise definitions of rotations, reflections and translations.

Provide real-world examples of rigid motions (e.g. Ferris wheels for rotation; mirrors for reflection; moving vehicles for translation).

Use graph paper, transparencies, tracing paper or dynamic geometry software to obtain images of a given figure under specified transformations.

Provide students with a pre-image and a final, transformed image, and ask them to describe the steps required to generate the final image. Show examples with more than one answer (e.g., a reflection might result in the same image as a translation).

Work backwards to determine a sequence of transformations that will carry (map) one figure onto another of the same size and shape.

Focus attention on the attributes (e.g. distances or angle measures) of a geometric figure that remain constant under various transformations.

Make the transition from transformations as physical motions to functions that take points in the plane as inputs and give other points as outputs. The correspondence between the initial and final points determines the transformation.

Analyze various figures (e.g. regular polygons, folk art designs or product logos) to determine which rotations and reflections carry (map) the figure onto itself. These transformations are the “symmetries” of the figure.

Emphasize the importance of understanding a transformation as the correspondence between initial and final points, rather than the physical motion.

Use a variety of means to represent rigid motions, including physical manipulatives, coordinate methods, and dynamic geometry software.

Common Misconceptions: G.CO.1-5

The terms “mapping” and “under” are used in special ways when studying transformations. A translation is a type of transformation that moves all the points in the object in a straight line in the same direction.

Students should know that not every transformation is a translation. Students sometimes confuse the terms “transformation” and “translation.”
Geometry: Congruence (G-CO)

Cluster: *Experiment with transformations in the plane.*

Standard: **G.CO.3** Given a rectangle, parallelogram, trapezoid, or regular polygon, describe the rotations and reflections that carry it onto itself.

Suggested Standards for Mathematical Practice (MP):
MP.3 Construct viable arguments and critique the reasoning of others.
MP.5 Use appropriate tools strategically.
MP.7 Look for and make use of structure.

Connections: See **G.CO.1**

Common Misconceptions: See **G.CO.1**

**Explanations and Examples: G.CO.3**

Describe and illustrate how a rectangle, parallelogram, and isosceles trapezoid are mapped onto themselves using transformations.

Calculate the number of lines of reflection symmetry and the degree of rotational symmetry of any regular polygon.

Students may use geometry software and/or manipulatives to model transformations.

**Examples:**

1. **Draw the shaded triangle after:**
   a. It has been translated −7 horizontally and +1 vertically.
      Label your answer *A*.
   b. It has been reflected over the x-axis.
      Label your answer *B*.
   c. It has been rotated 90° clockwise around the origin.
      Label your answer *C*.
   d. It has been reflected over the line *y* = *x*.
      Label your answer *D*.

2. **Describe fully the single transformation that:**
   a. Takes the shaded triangle onto the triangle labeled *E*.
   b. Takes the shaded triangle onto the triangle labeled *F*.

- For each of the following shapes, describe the rotations and reflections that carry it onto itself.

![Diagram](image)

**Instructional Strategies:** See **G.CO.1**
### Geometry: Congruence (G-CO)

**Cluster:** *Experiment with transformations in the plane.*

**Standard: G.CO.4** Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.

**Suggested Standards for Mathematical Practice (MP):**
- MP.5 Use appropriate tools strategically.
- MP.6 Attend to precision.
- MP.7 Look for and make use of structure.

**Connections:** See G.CO.1

**Common Misconceptions:** See G.CO.1

### Explanations and Examples: G.CO.4

Using previous comparisons and descriptions of transformations develop and understand the meaning of rotations, reflections, and translations based on angles, circles, perpendicular lines, parallel lines, and line segments.

Students may use geometry software and/or manipulatives to model transformations. Students may observe patterns and develop definitions of rotations, reflections, and translations.

**Examples:**
- Perform a rotation, reflection, and translation with a given polygon and give a written explanation of how each step meets the definitions of each transformation using correct mathematical terms.

- Construct the reflection definition by connecting any point on the preimage to its corresponding point on the reflected image and describe the line segment’s relationship to the line of reflection (e.g., the line of reflection is the perpendicular bisector of the segment).

- Construct the translation definition by connecting any point on the preimage to its corresponding point on the translated image, and connect a second point on the preimage to its corresponding point on the translated image, and describe how the two segments are equal in length, point in the same direction, and are parallel.

- Construct the rotation definition by connecting the center of rotation to any point on the preimage and to its corresponding point on the rotated image, and describe the measure of the angle formed and the equal measures of the segments that formed the angle as part of the definition.

**Instructional Strategies:** See G.CO.1

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G.CO.4
Geometry: Congruence *(G-CO)*

Cluster: *Experiment with transformations in the plane.*

**Standard: G.CO.5**  Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

**Suggested Standards for Mathematical Practice (MP):**
- **MP.3** Construct viable arguments and critique the reasoning of others.
- **MP.5** Use appropriate tools strategically.
- **MP.7** Look for and make use of structure.

**Connections:** See [G.CO.1](#)

**Common Misconceptions:** See [G.CO.1](#)

**Explanations and Examples: G.CO.5**

Transform a geometric figure given a rotation, reflection, or translation using graph paper, tracing paper, or geometry software.

Create sequences of transformations that map a geometric figure on to itself and another geometric figure.

Draw a specific transformation when given a geometric figure and a rotation, reflection or translation.

Predict and verify the sequence of transformations (a composition) that will map a figure onto another.

Students may use geometry software and/or manipulatives to model transformations and demonstrate a sequence of transformations that will carry a given figure onto another.

**Examples:**

- The triangle in the upper left of the figure below has been reflected across a line into the triangle in the lower right of the figure. Use a straightedge and compass to construct the line across which the triangle was reflected.

![Diagram](image-url)  

*Continued on next page*
Explanations and Examples: G.CO.5

- For the diagram below, describe the sequence of transformations that was used to carry $\triangle JKL$ (Image 1) onto Image 2.

Instructional Strategies: See G.CO.1
Geometry: Congruence (G-CO)

Cluster: Understand congruence in terms of rigid motion.

Standard: G.CO.6 Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.

Suggested Standards for Mathematical Practice (MP):
MP.3 Construct viable arguments and critique the reasoning of others.
MP.5 Use appropriate tools strategically. MP.7 Look for and make use of structure.

Connections: G.CO.6-8
An understanding of congruence using physical models, transparencies or geometry software is developed in Grade 8, and should be built upon in high school with greater attention to precise definitions, careful statements and proofs of theorems and formal reasoning.

Explanations and Examples: G.CO.6
For standards G.CO.6-8 the focus is for students to understand that rigid motions are at the foundation of the definition of congruence. Students reason from the basic properties of rigid motions (that they preserve distance and angle), which are assumed without proof. Rigid motions and their assumed properties can be used to establish the usual triangle congruence criteria, which can then be used to prove other theorems.

Use descriptions of rigid motion and transformed geometric figures to predict the effects rigid motion has on figures in the coordinate plane.

Knowing that rigid transformations preserve size and shape or distance and angle, use this fact to connect the idea of congruency and develop the definition of congruent. Determine if two figures are congruent by determining if rigid motions will turn one figure into the other.

Examples:
- $\triangle ABC$ has vertices $A(-1, 0), B(4, 0), C(2, 6)$
  a. Draw $\triangle ABC$ on the coordinate grid provided.
  b. Translate $\triangle ABC$ using the rule $(x, y) \rightarrow (x - 6, y - 5)$ to create $\triangle A'B'C'$. Record the new coordinate grid (using a different color if possible).
  $A'$ ____________, $B'$ ____________, $C'$ ____________
  c. Rotate $\triangle A'B'C'$ 90°CCW using the rule $(x, y) \rightarrow$ ____________ to create $\triangle A''B''C''$. Record the new coordinates below and add the triangle to your coordinate grid (using a different color if possible).
  $A''$ ____________, $B''$ ____________, $C''$ ____________
  d. Write ONE rule below that would change $\triangle ABC$ to $\triangle A''B''C''$ in one step.

Continued on next page
**Explanations and Examples: G.CO.6**

- Determine if the figures below are congruent. If so tell what rigid motions were used.

![Figure](image)

**Instructional Strategies: G.CO.6-8**

Develop the relationship between transformations and congruency. Allow adequate time and provide hands-on activities for students to visually and physically explore rigid motions and congruence.

Use graph paper, tracing paper or dynamic geometry software to obtain images of a given figure under specified rigid motions. Note that size and shape are preserved.

Use rigid motions (translations, reflections and rotations) to determine if two figures are congruent. Compare a given triangle and its image to verify that corresponding sides and corresponding angles are congruent.

Work backwards – given two figures that have the same size and shape, find a sequence of rigid motions that will map one onto the other.

Build on previous learning of transformations and congruency to develop a formal criterion for proving the congruency of triangles. Construct pairs of triangles that satisfy the ASA, SAS or SSS congruence criteria, and use rigid motions to verify that they satisfy the definition of congruent figures. Investigate rigid motions and congruence both algebraically (using coordinates) and logically (using proofs).

**Common Misconceptions: G.CO.6-8**

Some students may believe:

That combinations such as SSA or AAA are also a congruence criterion for triangles. Provide counterexamples for this misconception.

That all transformations, including dilations, are rigid motions. Provide counterexamples for this misconception.

That any two figures that have the same area represent a rigid transformation. Students should recognize that the areas remain the same, but preservation of side and angle lengths determine that the transformation is rigid.

That corresponding vertices do not have to be listed in order; however, it is useful to stress the importance of listing corresponding vertices in the same order so that corresponding sides and angles can be easily identified and that included sides or angles are apparent.
Geometry: Congruence *(G-CO)*

Cluster: *Understand congruence in terms of rigid motion.*

Standard: **G.CO.7** Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent.

Suggested Standards for Mathematical Practice (MP):
MP.3 Construct viable arguments and critique the reasoning of others.

Connections: See **G.CO.6**

**Explanations and Examples: G.CO.7**

A rigid motion is a transformation of points in space consisting of a sequence of one or more translations, reflections, and/or rotations. Rigid motions are assumed to preserve distances and angle measures. Two triangles are said to be congruent if one can be exactly superimposed on the other by a rigid motion, and the congruence theorems specify the conditions under which this can occur.

Students identify corresponding sides and corresponding angles of congruent triangles of congruent triangles. Explain that in a pair of congruent triangles, corresponding sides are congruent (distance is preserved) and corresponding angles are congruent (angles measure is preserved).

Demonstrate that when distance is preserved (corresponding sides are congruent) and angle measure is preserved (corresponding angles are congruent) the triangles must also be congruent.

**Examples:**
- How many ways can you construct a triangle congruent to the given triangle inside the rectangle? Demonstrate each.
Explanations and Examples: G.CO.7

- Are the following triangles congruent? Explain how you know.

Instructional Strategies: G.CO.6-8

Develop the relationship between transformations and congruency. Allow adequate time and provide hands-on activities for students to visually and physically explore rigid motions and congruence.

Use graph paper, tracing paper or dynamic geometry software to obtain images of a given figure under specified rigid motions. Note that size and shape are preserved.

Use rigid motions (translations, reflections and rotations) to determine if two figures are congruent. Compare a given triangle and its image to verify that corresponding sides and corresponding angles are congruent.

Work backwards – given two figures that have the same size and shape, find a sequence of rigid motions that will map one onto the other.

Build on previous learning of transformations and congruency to develop a formal criterion for proving the congruency of triangles. Construct pairs of triangles that satisfy the ASA, SAS or SSS congruence criteria, and use rigid motions to verify that they satisfy the definition of congruent figures. Investigate rigid motions and congruence both algebraically (using coordinates) and logically (using proofs).

Common Misconceptions: G.CO.6-8

Some students may believe:

That combinations such as SSA or AAA are also a congruence criterion for triangles. Provide counterexamples for this misconception.

That all transformations, including dilations, are rigid motions. Provide counterexamples for this misconception.

That any two figures that have the same area represent a rigid transformation. Students should recognize that the areas remain the same, but preservation of side and angle lengths determine that the transformation is rigid.

That corresponding vertices do not have to be listed in order; however, it is useful to stress the importance of listing corresponding vertices in the same order so that corresponding sides and angles can be easily identified and that included sides or angles are apparent.
**Cluster:** Understand congruence in terms of rigid motion.

**Standard:** G.CO.8 Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions.

**Suggested Standards for Mathematical Practice (MP):**
- MP.2 Reason abstractly and quantitatively.
- MP.3 Construct viable arguments and critique the reasoning of others.

**Connections:** See G.CO.6

**Common Misconceptions:** See G.CO.6

**Explanations and Examples: G.CO.8**

List the sufficient conditions to prove triangles are congruent.
Map a triangle with one of the sufficient conditions (e.g., SSS) onto the original triangle and show that corresponding sides and corresponding angles are congruent.

**Examples:**
- Josh is told that two triangles $ABC$ and $DEF$ share two sets of congruent sides and one pair of congruent angles: $AB$ is congruent to $DE$, $BC$ is congruent to $E$, and angle $C$ is congruent to angle $F$.
  He is asked if these two triangles must be congruent.
  Josh draws the two triangles below and says, “They are definitely congruent because they share all three side lengths”!
  
  o Explain Josh’s reasoning using one of the triangle congruence criteria: ASA, SSS, SAS.

  o Give an example of two triangles $ABC$ and $DEF$, fitting the criteria of this problem, which are not congruent.

![Diagram](Diagram.png)

*Continued on next page*
Explanations and Examples: G.CO.8

Sample Response:

a. Josh’s reasoning is incorrect because he has made the unwarranted assumption that angles C and F are right angles. However, with that additional assumption his statement is correct, since we may apply the Pythagorean theorem to conclude that $|AC|^2 = |AB|^2 - |BC|^2$ and $|DF|^2 = |DE|^2 - |EF|^2$.

Since $DE$ is congruent to $AB$ and $EF$ is congruent to $BC$ by hypothesis we can conclude that $AC$ must be congruent to $DF$ and so, by SSS, triangle $ABC$ is congruent to triangle $DEF$. Instead of SSS, we could also apply SAS using right angles $C$ and $F$ along with sides $AC$ and $BC$ for triangle $ABC$ and sides $DF$ and $EF$ for triangle $DEF$.

b. The information given amounts to SSA, two congruent sides and a congruent angle which is not the angle determined by the two sets of congruent sides. This is a lot of information and, as might be expected, does not leave much ambiguity. Consider five points $A$, $B$, $C$, $D$, $E$ as pictured below with isosceles triangle $ADE$:

Triangles $ABD$ and $ABE$ share angle $B$ and side $AB$ while $AD$ is congruent to $AE$ by construction. The triangles $ABD$ and $ABE$ are definitely not congruent, however, as one of them is properly contained within the other.

The given information heavily restricted this construction but we were still able to find two non congruent triangles sharing two congruent sides and a non-included congruent angle.

• Decide whether there is enough information to prove that the two shaded triangles are congruent. In the figure below, $ABCD$ is a parallelogram.

Solution: The two triangles are congruent by SAS. We have $AX \cong CX$ and $DX \cong BX$ since the diagonals of a parallelogram bisect each other and $\angle AXD \cong \angle BXC$ since they are vertical angles. Alternatively, we could use and argue via ASA. We have the opposite interior angles $\angle DAX \cong \angle BCX$ and $\angle ADX \cong \angle CBX$ and $AD \cong BC$ since opposite sides of a parallelogram are congruent.

Instructional Strategies: See G.CO.6

Build on previous learning of transformations and congruency to develop a formal criterion for proving the congruency of triangles. Construct pairs of triangles that satisfy the ASA, SAS or SSS congruence criteria, and use rigid motions to verify that they satisfy the definition of congruent figures. Investigate rigid motions and congruence both algebraically (using coordinates) and logically (using proofs).
Geometry: Congruence (G-CO)

Cluster: Prove geometric theorems.

Standard: G.CO.9  Prove theorems about lines and angles. Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment’s endpoints.

Suggested Standards for Mathematical Practice (MP):
MP.2  Reason abstractly and quantitatively.  MP.5  Use appropriate tools strategically.
MP.3  Construct viable arguments and critique the reasoning of others.

Connections: G.CO.9-11
Properties of lines and angles, triangles and parallelograms were investigated in Grades 7 and 8. In high school, these properties are revisited in a more formal setting, giving greater attention to precise statements of theorems and establishing these theorems by means of formal reasoning.

The theorem about the midline of a triangle can easily be connected to a unit on similarity. The proof of it is usually based on the similarity property that corresponding sides of similar triangles are proportional.

Explanations and Examples: G.CO.9
Identify and use the properties of congruence and equality (reflexive, symmetric, transitive) in proofs.
Order statements based on the Law of Syllogism when constructing a proof.
Interpret geometric diagrams by identifying what can and cannot be assumed.
Students may use geometric simulations (computer software or graphing calculator) to explore theorems about lines and angles.

Examples:
- The diagram below depicts the construction of a parallel line, above the ruler.
  The steps in the construction result in a line through the given point that is parallel to the given line.
  Which statement below justifies why the constructed line is parallel to the given line?

  a. When two lines are each perpendicular to a third line, the lines are parallel.
  b. When two lines are each parallel to a third line, the lines are parallel.
  c. When two lines are intersected by a transversal and alternate interior angles are congruent, the lines are parallel.
  d. When two lines are intersected by a transversal and corresponding angles are congruent, the lines are parallel.  (Correct answer d.)

  Continued on next page
Explanations and Examples: G.CO.9

- Prove that $\angle HIB \cong \angle DIG$, given that $\overline{AB} \parallel \overline{DE}$.

Instructional Strategies: G.CO.9-11

Encourage multiple ways of writing proofs, such as in narrative paragraphs, using flow diagrams, in two-column format, and using diagrams without words. Students should be encouraged to focus on the validity of the underlying reasoning while exploring a variety of formats for expressing that reasoning. Implementation of G.CO.10 may be extended to include concurrence of perpendicular bisectors and angle bisectors as preparation for G.C.3.

Classroom teachers and mathematics education researchers agree that students have a hard time learning how to do geometric proofs. An article by Battista and Clements (1995) (http://investigations.terc.edu/library/bookpapers/geometryand_proof.cfm) provides information for teachers to help students who struggle learn to do proof. The most significant implication for instructional strategies for proof is stated in their conclusion.

“Itenally, the most effective path to engendering meaningful use of proof in secondary school geometry is to avoid formal proof for much of students’ work. By focusing instead on justifying ideas while helping students build the visual and empirical foundation for higher levels of geometric thought, we can lead students to appreciate the need for formal proof. Only then will we be able to use it meaningfully as a mechanism for justifying ideas.”

The article and ideas from Niven (1987) offers a few suggestions about teaching proof in geometry:

- Initial geometric understandings and ideas should be taught “without excessive emphasis on rigor.” Develop basic geometric ideas outside an axiomatic framework, and then let the importance of the framework (and the framework itself) emerges from the geometry.
- Geometry is visual and should be taught in ways that leverage this aspect. Sketching, drawing and constructing figures and relationships between geometric objects should be central to any geometric study and certainly to proof. Battista and Clement make a powerful argument that the use of dynamic geometry software can be an important tool for helping students understand proof.
- “Avoid the deadly elaboration of the obvious” (Niven, p. 43). Often textbooks begin the treatment of formal proof with “easy” proofs, which appear to students to need no proof at all. After presenting many opportunities for students to “justify” properties of geometric figures, formal proof activities should begin with non-obvious conjectures.
- Use the history of geometry and real-world applications to help students develop conceptual understandings before they begin to use formal proof.

Continued on next page
**Instructional Strategies: G.CO.9-11**

Proofs in high school geometry should not be restricted to the two-column format. Most proofs at the college level are done in paragraph form, with the writer explaining and defending a conjecture. In many cases, the two-column format can hinder the student from making sense of the geometry by paying too much attention to format rather than mathematical reasoning.

Some of the theorems listed in this cluster (e.g. the ones about alternate interior angles and the angle sum of a triangle) are logically equivalent to the Euclidean parallel postulate, and this should be acknowledged.

Use dynamic geometry software to allow students to make conjectures that can, in turn, be formally proven. For example, students might notice that the base angles of an isosceles triangle always appear to be congruent when manipulating triangles on the computer screen and could then engage in a more formal discussion of why this occurs.

Common Core Standards Appendix A states, “Encourage multiple ways of writing proofs, such as in narrative paragraphs, using flow diagrams, in two-column format, and using diagrams without words. Students should be encouraged to focus on the validity of the underlying reasoning while exploring a variety of formats for expressing that reasoning” (p. 29). Different methods of proof will appeal to different learning styles in the classroom.

**Common Misconceptions: G.CO.9-11**

Research over the last four decades suggests that student misconceptions about proof abound:

- even after proving a generalization, students believe that exceptions to the generalization might exist;
- one counterexample is not sufficient;
- the converse of a statement is true (parallel lines do not intersect, lines that do not intersect are parallel); and
- a conjecture is true because it worked in all examples that were explored.

Each of these misconceptions needs to be addressed, both by the ways in which formal proof is taught in geometry and how ideas about “justification” are developed throughout a student’s mathematical education.
### Geometry: Congruence (G-CO)

**Cluster:** Prove geometric theorems.

**Standard: G.CO.10** Prove theorems about triangles. *Theorems include: measures of interior angles of a triangle sum to 180°; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.*

**Suggested Standards for Mathematical Practice (MP):**
- MP.2 Reason abstractly and quantitatively.
- MP.5 Use appropriate tools strategically.
- MP.3 Construct viable arguments and critique the reasoning of others.

**Connections:** See G.CO.9

**Common Misconceptions:** See G.CO.9

#### Explanations and Examples: G.CO.10

Order statements based on the Law of Syllogism when constructing a proof. Interpret geometric diagrams by identifying what can and cannot be assumed. Students may use geometric simulations (computer software or graphing calculator) to explore theorems about triangles.

**Examples:**
- For items 1 and 2, what additional information is required in order to prove the two triangles are congruent using the provided justification?
  - Use the set of choices in the box below. Select a side or angle and place it in the appropriate region. Only one side or angle can be placed in each region.

<table>
<thead>
<tr>
<th>AB</th>
<th>AC</th>
<th>AD</th>
<th>BC</th>
</tr>
</thead>
<tbody>
<tr>
<td>BD</td>
<td>CD</td>
<td>CE</td>
<td>DE</td>
</tr>
<tr>
<td>∠ABC</td>
<td>∠ABD</td>
<td>∠ACB</td>
<td>∠ADB</td>
</tr>
<tr>
<td>∠BAC</td>
<td>∠CDE</td>
<td>∠CED</td>
<td>∠DCE</td>
</tr>
</tbody>
</table>

**Item 1**

[Diagram showing ASA Postulate]

**Key:** Item 1 – ∠ABD ≅ ∠ABC

**Item 2**

[Diagram showing SAS Theorem]

**Key:** Item 2 – AC ≅ CE

Continued on next page
Explanations and Examples: G.CO.10

- Given that $\triangle ABC$ is isosceles, prove that $\angle ABC \cong \angle ACB$.

Instructional Strategies: See G.CO.9
Geometry: Congruence (G-CO)

Cluster: Prove geometric theorems.

Standard: G.CO.11  Prove theorems about parallelograms. *Theorems include:* opposite sides are congruent, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and conversely, rectangles are parallelograms with congruent diagonals.

Suggested Standards for Mathematical Practice (MP):

- MP.2  Reason abstractly and quantitatively.
- MP.5  Use appropriate tools strategically.
- MP.3  Construct viable arguments and critique the reasoning of others.

Connections: See G.CO.9

Common Misconceptions: See G.CO.9

Explanations and Examples: G.CO.11

Students may use geometric simulations (computer software or graphing calculator) to explore theorems about parallelograms.

Examples:

- Suppose that $ABCD$ is a parallelogram, and that $M$ and $N$ are the midpoints of $AB$ and $CD$, respectively.
  
  Prove that $MN \equiv AD$, and that the line $MN$ is parallel to $AD$.

**Solution:**

The diagram above consists of the given information, and one additional line segment, $MD$, which we will use to demonstrate the result. We claim that triangles $\triangle AMD$ and $\triangle NDM$ are congruent by SAS:

- We have $MD \equiv DM$ by reflexitivity.
- We have $\angle AMD = \angle NDM$ since they are opposite interior angles of the transversal $MD$ through parallel lines $AB$ and $CD$.
- We have $MA = ND$, since $M$ and $N$ are midpoints of their respective sides, and opposite sides of parallelograms are congruent $MA = \frac{1}{2}(AB) = \frac{1}{2}(CD) = ND$.

Now since corresponding parts of congruent triangles are congruent, we have $DA \equiv NM$, as desired. Similarly, we have congruent opposite interior angles $\angle DMN \equiv \angle MDA$, so $MN$ is parallel to $AD$.

Instructional Strategies: See G.CO.9
Geometry: Congruence *(G-CO)*

**Cluster:** Make geometric constructions. *(Formalize and explain processes.)*

**Standard:** G.CO.12  Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.). *Copying a segment; copying an angle; bisecting a segment; bisecting an angle; constructing perpendicular lines, including the perpendicular bisector of a line segment; and constructing a line parallel to a given line through a point not on the line.*

**Suggested Standards for Mathematical Practice (MP):**
MP.5 Use appropriate tools strategically.  
MP.6 Attend to precision.

**Connections:** G.CO.12-13  
Drawing geometric shapes with rulers, protractors and technology is developed in Grade 7. In high school, students perform formal geometry constructions using a variety of tools. Students will utilize proofs to justify validity of their constructions.

**Explanations and Examples: G.CO.12**

For standards G.CO.12-13, the expectation is to build on prior student experience with simple constructions. Emphasize the ability to formalize and explain how these constructions result in the desired objects. Some of these constructions are closely related to previous standards and can be introduced with them.

Students may use geometric software to make geometric constructions.

**Examples:**

- Construct a triangle given the lengths of two sides and the measure of the angle between the two sides.
- Construct the circumcenter of a given triangle.
- Construct the perpendicular bisector of a line segment.
  
  This construction can also be used to construct a 90 degree angle or to find the midpoint of a line.

  1. Mark two points on your line, A and B - this construction will give you a straight line which passes exactly half way between these two points and is perpendicular (at right angles) to the line.
  2. Open your compasses to a distance more than half way between A and B.
  3. With the point of the compass on one of the points, draw circular arcs above and below the line, at P and Q.
  4. Keeping the compasses set to exactly the same distance, repeat with the compass point on your other point.
  5. Draw a line through the P and Q.
  6. PQ is the perpendicular bisector of AB - check that the angles are exactly 90 degrees and that it does indeed halve the distance between A and B.

[http://motivate.maths.org/content/accurate-constructions](http://motivate.maths.org/content/accurate-constructions)

*Continued on next page*
Explanations and Examples: G.CO.12

- You have been asked to place a warehouse so that it is an equal distance from the three roads indicated on the following map. Find this location and show your work.

- Show how to fold your paper to physically construct this point as an intersection of two creases.
- Explain why the above construction works and, in particular, why you only needed to make two creases.

Solution: (This task connects to standard G.C.3)

1. Fold and crease the paper so that Oak lies on top of Rio. Do the same so that Oak lies on top of Elm. The point of intersection of the two creases is the point an equal distance from the three sides.

2. Since the desired location should be an equal distance from three sides of triangle ABC, we are looking for the center of the circle inscribed in the triangle. The center of the inscribed circle, called the incenter, can be found by constructing the angle bisectors of the three interior angles of the triangle, as in the diagram below. Since these angle bisectors are concurrent, it is sufficient to construct two of the angle bisectors (and hence only make two creases in part (a)).

Now we show the concurrence of the three angle bisectors: It is easy to see that the distance from the warehouse \( W = WH \) to Rio equals the distance from \( W \) to Oak. Namely, draw perpendiculars from \( W \) to both Rio and Oak, with respective intersection points \( X \) and \( Y \).

The triangles \( \triangle WXC \) and \( \triangle WYC \) are congruent since they are right triangles with \( \angle WCX = \angle WCY \) and sharing side \( WC \). So \( WX = WY \). Similarly, drawing a perpendicular to Elm through \( W \) meeting Elm at \( Z \), we have \( WY = WZ \). Combining the two equalities, we learn that \( WX = WZ \), so that \( W \) is on the angle bisector and the three angle bisectors are concurrent.

Continued on next page
**Instructional Strategies: G.CO.12-13**

Students should analyze each listed construction in terms of what simpler constructions are involved (e.g., constructing parallel lines can be done with two different constructions of perpendicular lines).

Using congruence theorems, ask students to prove that the constructions are correct.

Provide meaningful problems (e.g. constructing the centroid or the incenter of a triangle) to offer students practice in executing basic constructions.

Challenge students to perform the same construction using a compass and string. Use paper folding to produce a reflection; use bisections to produce reflections.

Ask students to write “how-to” manuals, giving verbal instructions for a particular construction. Offer opportunities for hands-on practice using various construction tools and methods.

Compare dynamic geometry commands to sequences of compass-and-straightedge steps. Prove, using congruence theorems, that the constructions are correct.

**Common Misconceptions: G.CO.12-13**

Some students may believe that a construction is the same as a sketch or drawing. Emphasize the need for precision and accuracy when doing constructions. Stress the idea that a compass and straightedge are identical to a protractor and ruler. Explain the difference between measurement and construction.
Geometry: Congruence (G-CO)

Cluster: Make geometric constructions. (Formalize and explain processes.)

Standard: G.CO.13 Construct an equilateral triangle, a square, and a regular hexagon inscribed in a circle.

Suggested Standards for Mathematical Practice (MP):
MP.5 Use appropriate tools strategically. MP.6 Attend to precision.

Connections: See G.CO.12 Common Misconceptions: See G.CO.12

Explanations and Examples: G.CO.13
For standards G.CO.12-13, the expectation is to build on prior student experience with simple constructions. Emphasize the ability to formalize and explain how these constructions result in the desired objects. Some of these constructions are closely related to previous standards and can be introduced with them. Students may use geometric software to make geometric constructions.

Examples:
- Construct a regular hexagon inscribed in a circle.
  This construction can also be used to draw a 120° angle.
  Keep your compasses to the same setting throughout this construction.
  Draw a circle.
  Mark a point, P, on the circle.
  Put the point of your compasses on P and draw arcs to cut the circle at Q and U.
  Put the point of your compasses on Q and draw an arc to cut the circle at R.
  Repeat with the point of the compasses at R and S to draw arcs at S and T.
  Join PQRSTU to form a regular hexagon.
  Measure the lengths to check they are all equal, and the angles to check they are all 120 degrees.

  http://motivate.maths.org/content/accurate-constructions

- Find two ways to construct a hexagon inscribed in a circle as shown.

Instructional Strategies: See G.CO.12
Cluster: *Understand similarity in terms of similarity transformations.*

**Standard: G.SRT.1** Verify experimentally the properties of dilations given by a center and a scale factor:

a. A dilation takes a line not passing through the center of the dilation to a parallel line, and leaves a line passing through the center unchanged.

b. The dilation of a line segment is longer or shorter in the ratio given by the scale factor.

**Suggested Standards for Mathematical Practice (MP):**

MP.2 Reason abstractly and quantitatively.  
MP.5 Use appropriate tools strategically.  
MP.6 Attend to precision. 
MP.8 Look for and express regularity in repeated reasoning.

**Connections: G.SRT.1-3**

Dilations and similarity, including the AA criterion, are investigated in Grade 8, and these experiences should be built upon in high school with greater attention to precise definitions, careful statements and proofs of theorems and formal reasoning.

**Explanations and Examples: G.SRT.1**

Students should understand that a dilation is a transformation that moves each point along the ray through the point emanating from a fixed center, and multiplies distances from the center by a common scale factor. Perform a dilation with a given center and scale factor on a figure in the coordinate plane. Verify that when a side passes through the center of dilation, the side and its image lie on the same line. Verify that corresponding sides of the preimage and images are parallel. Verify that a side length of the image is equal to the scale factor multiplied by the corresponding side length of the preimage.

Students may use geometric simulation software to model transformations. Students may observe patterns and verify experimentally the properties of dilations.

**Examples:**

- Suppose we apply a dilation by a factor of 2, centered at the point $P$ to the figure below.

```
A ---- B ---- C
  `----> l
```

a. In the picture, locate the images $A'$, $B'$, and $C'$ of the points $A$, $B$, $C$ under this dilation.

b. Based on your picture in part a., what do you think happens to the line $l$ when we perform the dilation?

c. Based on your picture in part a., what appears to be the relationship between the distance $A'B'$ and the distance $AB$?

d. Can you prove your observations in part c?

*Continued on next page*
Explanations and Examples: G.SRT.1

- Draw a polygon. Pick a point and construct a dilation of the polygon with that point as the center. Identify the scale factor that you used.

**Example Response:**

![Diagram of a dilation]

Instructional Strategies: G.SRT.1-3

Allow adequate time and hands-on activities for students to explore dilations visually and physically.

Use graph paper and rulers or dynamic geometry software to obtain images of a given figure under dilations having specified centers and scale factors. Carefully observe the images of lines passing through the center of dilation and those not passing through the center, respectively. A line segment passing through the center of dilation will simply be shortened or elongated but will lie on the same line, while the dilation of a line segment that does not pass through the center will be parallel to the original segment (this is intended as a clarification of Standard 1a).

Illustrate two-dimensional dilations using scale drawings and photocopies.

Measure the corresponding angles and sides of the original figure and its image to verify that the corresponding angles are congruent and the corresponding sides are proportional (i.e. stretched or shrunk by the same scale factor). Investigate the SAS and SSS criteria for similar triangles.

Use graph paper and rulers or dynamic geometry software to obtain the image of a given figure under a combination of a dilation followed by a sequence of rigid motions (or rigid motions followed by dilation).

Work backwards – given two similar figures that are related by dilation, determine the center of dilation and scale factor. Given two similar figures that are related by a dilation followed by a sequence of rigid motions, determine the parameters of the dilation and rigid motions that will map one onto the other.

Using the theorem that the angle sum of a triangle is 180°, verify that the AA criterion is equivalent to the AAA criterion. Given two triangles for which AA holds, use rigid motions to map a vertex of one triangle onto the corresponding vertex of the other in such a way that their corresponding sides are in line. Then show that dilation will complete the mapping of one triangle onto the other.

Students may be interested in scale models or experiences with blueprints and scale drawings (perhaps in a work related situation) to illustrate similarity.

*Continued on next page*
Common Misconceptions: G.SRT.1-3

Some students often do not recognize that congruence is a special case of similarity. Similarity with a scale factor equal to 1 becomes a congruency.

Students may not realize that similarities preserve shape, but not size. Angle measures stay the same, but side lengths change by a constant scale factor.

Students may incorrectly apply the scale factor. For example, students will multiply instead of divide with a scale factor that reduces a figure or divide instead of multiply when enlarging a figure.

Some students often do not list the vertices of similar triangles in order. However, the order in which vertices are listed is preferred and especially important for similar triangles so that proportional sides can be correctly identified.
Cluster: Understand similarity in terms of similarity transformations.

Standard: G.SRT.2 Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides.

Suggested Standards for Mathematical Practice (MP):
MP.3 Construct viable arguments and critique the reasoning of others.
MP.5 Use appropriate tools strategically.
MP.7 Look for and make use of structure.

Connections: See G.SRT.1

Common Misconceptions: See G.SRT.1

Explanations and Examples: G.SRT.2

Use the idea of dilation transformations to develop the definition of similarity. Understand that a similarity transformation is a rigid motion followed by a dilation.

Demonstrate that in a pair of similar triangles, corresponding angles are congruent (angle measure is preserved) and corresponding sides are proportional.

Determine that two figures are similar by verifying that angle measure is preserved and corresponding sides are proportional.

Students may use geometric simulation software to model transformations and demonstrate a sequence of transformations to show congruence or similarity of figures.

Examples:

- Are these two figures similar? Explain why or why not.

Continued on next page
Explanations and Examples: G.SRT.2

- In the picture below, line segments $AD$ and $BC$ intersect at $X$. Line segments $AB$ and $CD$ are drawn, forming two triangles $\triangle AXB$ and $\triangle CXD$.

In each part a-d below, some additional assumptions about the picture are given. In each problem, determine whether the given assumptions are enough to prove that the two triangles are similar, and if so, what the correct correspondence of vertices is. If the two triangles must be similar, prove this result by describing a sequence of similarity transformations that maps one variable to the other. If not explain why not.

a. The lengths of $AX$ and $AD$ satisfy the equation $2AX = 3XD$.

b. The lengths $AX$, $BX$, $CX$, and $DX$ satisfy the equation $\frac{AX}{BX} = \frac{DX}{CX}$.

c. Lines $AB$ and $CD$ are parallel.

d. $\angle XAB$ is congruent to angle $\angle XCD$.

Solution:

a. We are given that $2AX = 3XD$. This is not enough information to prove similarity. To see that in a simple way draw an arbitrary triangle $\triangle AXB$. Extend $AX$ and choose a point D on the extended line so that $2AX=3XD$. Extend $BX$ and choose a point C on the extended line so that $2BX=XC$. Now triangles $\triangle AXB$ and $\triangle CXD$ satisfy the given conditions but are not similar. (If you are extremely unlucky, $\triangle AXB$ and $\triangle CXD$ might be similar by a different correspondence of sides. If this happens, rotate the line BC a little bit. The lengths of $AX$, $BX$, $XC$ remain the same but the triangles are no longer similar.)

b. We are given that $\frac{AX}{DX} = \frac{BX}{CX}$. Rearranging this proportion gives $\frac{AX}{DX} = \frac{BX}{CX}$. Let $k = \frac{AX}{DX}$. Suppose we rotate the triangle $\triangle DXC$ 180 degrees about point X, as in part (a), so that the angle $\angle DXC$ coincides with angle $\angle AXB$. Then dilate the triangle $\triangle DXC$ by a factor of $k$ about the center $X$. This dilation moves the point $D$ to $A$, since $k(DX) = AX$, and moves $C$ to $B$, since $k(CX) = BX$. Then since the dilation fixes $X$, and dilations take line segments to line segments, we see that the triangle $\triangle DXC$ is mapped to triangle $\triangle AXB$. So the original triangle $\triangle DXC$ is similar to $\triangle AXB$. (Note that we state the similarity so that the vertices of each triangle are written in corresponding order.)

c. Again, rotate triangle $\triangle DXC$ so that angle $\angle DXC$ coincides with angle $\angle AXB$. Then the image of the side $CD$ under this rotation is parallel to the original side $CD$, so the new side is still parallel to side $AB$. Now, apply a dilation about point $X$ that moves the vertex $C$ to point $B$. This dilation moves the line $CD$ to a line through $B$ parallel to the previous line $CD$. We already know that line $AB$ is parallel to $CD$, so the dilation must move the line $CD$ onto the line $AB$. Since the dilation moves $D$ to a point on the ray $XA$ and on the line $AB$, $D$ must move to $A$. Therefore, the rotation and dilation map the triangle $\triangle DXC$ to the triangle $\triangle AXB$, thus $\triangle DXC$ is similar to $\triangle AXB$.

d. Suppose we draw the bisector of angle $\angle AXC$, and reflect the triangle $\triangle CXD$ across this angle bisector. This maps the segment $XC$ onto the segment $XA$; and since reflections preserve angles, it also maps segment $XD$ onto segment $XB$. Since angle $\angle XCD$ is congruent to angle $\angle XAB$, we also know that the image of side $CD$ is parallel to side $AB$. Therefore, if we apply a dilation about the point $X$ that takes the new point $C$ to $A$, then the new line $CD$ will be mapped onto the line $AB$, by the same reasoning used in (c). Therefore, the new point $D$ is mapped to $B$, and thus the triangle $\triangle XCD$ is mapped to triangle $\triangle XAB$. So triangle $\triangle XCD$ is similar to triangle $\triangle XAB$. (Note that this is not the same correspondence we had in parts (b) and (c))

Instructional Strategies: See G.SRT.1
**Geometry: Similarity, Right Triangles, and Trigonometry (G-SRT)**

**Cluster:** *Understand similarity in terms of similarity transformations.*

**Standard:** **G.SRT.3** Use the properties of similarity transformations to establish the AA criterion for two triangles to be similar.

**Suggested Standards for Mathematical Practice (MP):**
MP.3 Construct viable arguments and critique the reasoning of others.

**Connections:** See **G.SRT.1**

**Common Misconceptions:** See **G.SRT.1**

**Explanations and Examples:** **G.SRT.3**
Show and explain that when two angle measures are known (AA), the third angle measure is also know (Third Angle Theorem).

Identify and explain that AA similarity is a sufficient condition for two triangles to be similar.

**Examples:**
- Are all right triangles similar to one another? How do you know?
- What is the least amount of information needed to prove two triangles are similar? How do you know?
- Using a ruler and a protractor, prove AA similarity.

**Instructional Strategies:** See **G.SRT.1**
Using the theorem that the angle sum of a triangle is 180°, verify that the AA criterion is equivalent to the AAA criterion. Given two triangles for which AA holds, use rigid motions to map a vertex of one triangle onto the corresponding vertex of the other in such a way that their corresponding sides are in line. Then show that dilation will complete the mapping of one triangle onto the other.
Geometry: Similarity, Right Triangles, and Trigonometry (G-SRT)

Cluster: Prove theorems involving similarity.

Standard: G.SRT.4 Prove theorems about triangles. Theorems include: a line parallel to one side of a triangle divides the other two proportionally, and conversely; the Pythagorean Theorem proved using triangle similarity.

Suggested Standards for Mathematical Practice (MP):
MP.3 Construct viable arguments and critique the reasoning of others.
MP.5 Use appropriate tools strategically.

Connections: G.SRT.4-5
The Pythagorean theorem and its converse are proved and applied in Grade 8. In high school, another proof, based on similar triangles, is presented.
The alternate interior angle theorem and its converse, as well as properties of parallelograms, are established informally in Grade 8 and proved formally in high school.

Explanations and Examples: G.SRT.4
Use AA, SAS, SSS similarity theorems to prove triangles are similar.
Use triangle similarity to prove other theorems about triangles
- Prove a line parallel to one side of a triangle divides the other two proportionally, and it’s converse
- Prove the Pythagorean Theorem using triangle similarity.

Students may use geometric simulation software to model transformations and demonstrate a sequence of transformations to show congruence or similarity of figures.

Examples:
- Prove that if two triangles are similar, then the ratio of corresponding attitudes is equal to the ratio of corresponding sides.
- How does the Pythagorean Theorem support the case for triangle similarity?
  - View the video below and create a visual proving the Pythagorean Theorem using similarity.
  http://www.youtube.com/watch?v=LrS5_l-gk94
- To prove the Pythagorean Theorem using triangle similarity:
  We can cut a right triangle into two parts by dropping a perpendicular onto the hypotenuse.
  Since these triangles and the original one have the same angles, all three are similar.
  Therefore
  \[
  \frac{x}{a} = \frac{a^2}{c}, \quad \frac{c-x}{b} = \frac{b^2}{c}
  \]
  \[
  x = \frac{a^2}{c}, \quad c-x = \frac{b^2}{c}
  \]
  \[
  x + (c-x) = c
  \]
  \[
  \frac{a^2}{c} + \frac{b^2}{c} = c
  \]
  \[
  a^2 + b^2 = c^2
  \]

Continued on next page
**Instructional Strategies: G.SRT.4-5**

Review triangle congruence criteria and similarity criteria, if it has already been established.
Review the angle sum theorem for triangles, the alternate interior angle theorem and its converse, and properties of parallelograms. Visualize it using dynamic geometry software.

Using SAS and the alternate interior angle theorem, prove that a line segment joining midpoints of two sides of a triangle is parallel to and half the length of the third side. Apply this theorem to a line segment that cuts two sides of a triangle proportionally.

Generalize this theorem to prove that the figure formed by joining consecutive midpoints of sides of an arbitrary quadrilateral is a parallelogram. (This result is known as the Midpoint Quadrilateral Theorem or Varignon’s Theorem.)

Use cardboard cutouts to illustrate that the altitude to the hypotenuse divides a right triangle into two triangles that are similar to the original triangle. Then use AA to prove this theorem. Then, use this result to establish the Pythagorean relationship among the sides of a right triangle ($a^2 + b^2 = c^2$) and thus obtain an algebraic proof of the Pythagorean Theorem.

Prove that the altitude to the hypotenuse of a right triangle is the geometric mean of the two segments into which its foot divides the hypotenuse.

Prove the converse of the Pythagorean Theorem, using the theorem itself as one step in the proof. Some students might engage in an exploration of Pythagorean Triples (e.g., 3-4-5, 5-12-13, etc.), which provides an algebraic extension and an opportunity to explore patterns.

**Common Misconceptions: G.SRT.4-5**

Some students may confuse the alternate interior angle theorem and its converse as well as the Pythagorean theorem and its converse.
Cluster: Prove theorems involving similarity.

Standard: G.SRT.5 Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

Suggested Standards for Mathematical Practice (MP):
MP.3 Construct viable arguments and critique the reasoning of others. MP.6 Attend to precision.
MP.4 Model with mathematics. MP.7 Look for and make use of structure.

Connections: See G.SRT.4 Common Misconceptions: See G.SRT.4

Explanations and Examples: G.SRT.5
Similarity postulates include SSS, SAS, and AA.
Congruence postulates include SSS, SAS, ASA, AAS, and H-L.
Apply triangle congruence and triangle similarity to solve problem situations (e.g., indirect measurement, missing sides/angle measures, side splitting).
Students may use geometric simulation software to model transformations and demonstrate a sequence of transformations to show congruence or similarity of figures.

Examples:
- This diagram is made up of four regular pentagons that are all the same size.

- Find the measure of \( \angle AEJ \)
- Find the measure of \( \angle BJF \)
- Find the measure of \( \angle KJM \)

Instructional Strategies: 8.G.6-8
Geometry: Similarity, Right Triangles, and Trigonometry (G-SRT)

Cluster: Define trigonometric ratios and solve problems involving right triangles.

Standard: G.SRT.6 Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.

Suggested Standards for Mathematical Practice (MP):
MP.2 Reason abstractly and quantitatively. MP.7 Look for and make use of structure.
MP.6 Attend to precision. MP.8 Look for and express regularity in repeated reasoning.

Connections: G.SRT.6-8
Trigonometry is not introduced until high school. Right triangle trigonometry (a geometry topic) has implications when studying algebra and functions. For example, trigonometric ratios are functions of the size of an angle, the trigonometric functions can be revisited after radian measure has been studied, and the Pythagorean theorem can be used to show that \((\sin A)^2 + (\cos A)^2 = 1\).

Explanations and Examples: G.SRT.6
Students may use applets to explore the range of values of the trigonometric ratios as \(\theta\) ranges from 0 to 90 degrees. Use the characteristics of similar figures to justify trigonometric ratios.

\[
\begin{align*}
\text{sine of } \theta &= \sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} \\
\text{cosine of } \theta &= \cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} \\
\text{tangent of } \theta &= \tan \theta = \frac{\text{opposite}}{\text{adjacent}} \\
\text{cosecant of } \theta &= \csc \theta = \frac{\text{hypotenuse}}{\text{opposite}} \\
\text{secant of } \theta &= \sec \theta = \frac{\text{hypotenuse}}{\text{adjacent}} \\
\text{cotangent of } \theta &= \cot \theta = \frac{\text{adjacent}}{\text{opposite}}
\end{align*}
\]

Examples:
- Find the sine, cosine, and tangent of \(x\).

![Diagram of a right triangle with sides 4 and 3, and an angle labeled \(x\).]
Explanations and Examples: G.SRT.6

- Explain why the sine of $x$ is the same regardless of which triangle is used to find it in the figure below.

![Diagram of a right triangle with an angle $x$ and sine labeled]

Instructional Strategies: G.SRT.6-8

Review vocabulary (opposite and adjacent sides, legs, hypotenuse and complementary angles) associated with right triangles.

Make cutouts or drawings of right triangles or manipulate them on a computer screen using dynamic geometry software and ask students to measure side lengths and compute side ratios. Observe that when triangles satisfy the AA criterion, corresponding side ratios are equal. Side ratios are given standard names, such as sine, cosine and tangent. Allow adequate time for students to discover trigonometric relationships and progress from concrete to abstract understanding of the trigonometric ratios.

Show students how to use the trigonometric function keys on a calculator. Also, show how to find the measure of an acute angle if the value of its trigonometric function is known.

Investigate sines and cosines of complementary angles, and guide students to discover that they are equal to one another. Point out to students that the “co” in cosine refers to the “sine of the complement.”

Observe that, as the size of the acute angle increases, sines and tangents increase while cosines decrease. Stress trigonometric terminology by the history of the word “sine” and the connection between the term “tangent” in trigonometry and tangents to circles.

Have students make their own diagrams showing a right triangle with labels showing the trigonometric ratios. Although students like mnemonics such as SOHCAHTOA, these are not a substitute for conceptual understanding. Some students may investigate the reciprocals of sine, cosine, and tangent to discover the other three trigonometric functions.

Use the Pythagorean theorem to obtain exact trigonometric ratios for 30°, 45°, and 60° angles. Use cooperative learning in small groups for discovery activities and outdoor measurement projects.

Have students work on applied problems and project, such as measuring the height of the school building or a flagpole, using clinometers and the trigonometric functions.

Common Misconceptions: G.SRT.6-8

Some students believe that right triangles must be oriented a particular way.

Some students do not realize that opposite and adjacent sides need to be identified with reference to a particular acute angle in a right triangle.

Some students believe that the trigonometric ratios defined in this cluster apply to all triangles, but they are only defined for acute angles in right triangles.
Geometry: Similarity, Right Triangles, and Trigonometry (G-SRT)

Cluster: Define trigonometric ratios and solve problems involving right triangles.

Standard: G.SRT.7 Explain and use the relationship between the sine and cosine of complementary angles.

Suggested Standards for Mathematical Practice (MP):
MP.2 Reason abstractly and quantitatively.
MP.3 Construct viable arguments and critique the reasoning of others.

Connections: See G.SRT.6
Common Misconceptions: See G.SRT.6

Explanations and Examples: G.SRT.7
Calculate sine and cosine ratios for acute angles in a right triangle when given two side lengths. Use a diagram of a right triangle to explain that for a pair of complimentary angles A and B, the sine of angle A is equal to the cosine of angle B and the cosine of angle A is equal to the sine of angle B.

Geometric simulation software applets and graphing calculators can be used to explore the relationship between sine and cosine.

Examples:
• What is the relationship between cosine and sine in relation to complementary angles?
  o Construct a table demonstrating the relationship between sine and cosine of complementary angles.

• Find the second acute angle of a right triangle given that the first acute angles has a measure of 39°.

• Complete the following statement: If \( \sin 30^\circ = \frac{1}{2} \), then the \( \cos______ = \frac{1}{2} \). 

• Find the sine and cosine of angle \( \theta \) in the triangle below. What do you notice?

Instructional Strategies: See G.SRT.6
### Geometry: Similarity, Right Triangles, and Trigonometry (G-SRT)

#### Cluster: Define trigonometric ratios and solve problems involving right triangles.

#### Standard: G.SRT.8

Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems. (★)

#### Suggested Standards for Mathematical Practice (MP):

MP.1 Make sense of problems and persevere in solving them.  
MP.4 Model with mathematics.  
MP.5 Use appropriate tools strategically.

#### Connections: See G.SRT.6  

#### Common Misconceptions: See G.SRT.6

#### Explanations and Examples: G.SRT.8

Use angle measures to estimate side lengths (e.g., The side across from a 33° angle will be shorter than the side across from a 57° angle).

Use side lengths to estimate angle measures (e.g., The angle opposite of a 10 cm side will be larger than the angle across from a 9 cm side).

Draw right triangles that describe real world problems and label the sides and angles with their given measures.

Solve application problems involving right triangles, including angle of elevation and depression, navigation, and surveying.

Students may use graphing calculators or programs, tables, spreadsheets, or computer algebra systems to solve right triangle problems.

#### Examples:

- Find the height of a flagpole to the nearest tenth if the angle of elevation of the sun is 28° and the shadow of the flagpole is 50 feet.

![Flagpole Diagram]

- A teenager whose eyes are 5 feet above ground level is looking into a mirror on the ground and can see the top of a building that is 30 feet away from the teenager. The angle of elevation from the center of the mirror to the top of the building is 75°. How tall is the building? How far away from the teenager’s feet is the mirror?

- While traveling across flat land, you see a mountain directly in front of you. The angle of elevation to the peak is 3.5°. After driving 14 miles closer to the mountain, the angle of elevation is 9°24′36″. Explain how you would set up the problem, and find the approximate height of the mountain.

#### Instructional Strategies: See G.SRT.6
Cluster: **Understand and apply theorems about circles.**

**Standard: G.C.1** Prove that all circles are similar.

**Suggested Standards for Mathematical Practice (MP):**
MP.3 Construct viable arguments and critique the reasoning of others.
MP.5 Use appropriate tools strategically.

**Connections: G.C.1-2**
The expectation is to emphasize the similarity of all circles.

**Explanations and Examples: G.C.1**
Using the fact that the ratio of diameter to circumference is the same for circles, prove that all circles are similar. Prove that all circles are similar by showing that for a dilation centered at the center of a circle, the preimage and the image have equal central angle measures. Students may use geometric simulation software to model transformations and demonstrate a sequence of transformations to show congruence or similarity of figures.

**Examples:**
- Show that the two given circles are similar by stating the necessary transformations from \( C \) to \( D \).
  - \( C \): center \((2, 3)\) radius 5
  - \( D \): center \((-1, 4)\) radius 10

- Draw or find examples of several different circles. In what ways are they related? How can you describe this relationship in terms of geometric ideas? Form a hypothesis and prove it.

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**Instructional Strategies: G.C.1-3**

Given any two circles in a plane, show that they are related by dilation. Guide students to discover the center and scale factor of this dilation and make a conjecture about all dilations of circles.

Starting with the special case of an angle inscribed in a semicircle, use the fact that the angle sum of a triangle is 180° to show that this angle is a right angle. Using dynamic geometry, students can grab a point on a circle and move it to see that the measure of the inscribed angle passing through the endpoints of a diameter is always 90°. Then extend the result to any inscribed angles. For inscribed angles, proofs can be based on the fact that the measure of an exterior angle of a triangle equals the sum of the measures of the nonadjacent angles. Consider cases of acute or obtuse inscribed angles.

Use properties of congruent triangles and perpendicular lines to prove theorems about diameters, radii, chords, and tangent lines.

Use formal geometric constructions to construct perpendicular bisectors of the sides and angle bisectors of a given triangle. Their intersections are the centers of the circumscribed and inscribed circles, respectively.

Dissect an inscribed quadrilateral into triangles, and use theorems about triangles to prove properties of these quadrilaterals and their angles.

Challenge students to generalize the results about angle sums of triangles and quadrilaterals to a corresponding result for n-gons.

**Common Misconceptions: G.C.1-3**

Students sometimes confuse inscribed angles and central angles. For example, they will assume that the inscribed angle is equal to the arc like a central angle.

Students may think they can tell by inspection whether a line intersects a circle in exactly one point. It may be beneficial to formally define a tangent line as the line perpendicular to a radius at the point where the radius intersects the circle.

Students may confuse the segment theorems. For example, they will assume that the inscribed angle is equal to the arc like a central angle.
Geometry: Circles (G-C)

Cluster: Understand and apply theorems about circles.

Standard: G.C.2 Identify and describe relationships among inscribed angles, radii, and chords. Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.

Suggested Standards for Mathematical Practice (MP):
- MP.1 Make sense of problems and persevere in solving them.
- MP.3 Construct viable arguments and critique the reasoning of others.
- MP.5 Use appropriate tools strategically.
- MP.6 Attend to precision.

Connections: See G.C.1

Common Misconceptions: See G.C.1

Explanations and Examples: G.C.2

Identify central angles, inscribed angles, circumscribed angles, diameters, radii, chords, and tangents.
Describe the relationship between a central angle and the arc it intercepts.
Describe the relationship between an inscribed angle and the arc it intercepts.
Describe the relationship between a circumscribed angle and the arcs it intercepts.
Recognize that an inscribed angle whose sides intersect the endpoints of the diameter of a circle is a right angle.
Recognize that the radius of a circle is perpendicular to the tangent where the radius intersects the circle.

Examples:

- Given the circle below with radius of 10 and chord length of 12, find the distance from the chord to the center of the circle.
Explanations and Examples: G.C.2

- Find the unknown length in the picture below.

![Diagram](image)

**Solution:**
The theorem for a secant segment and a tangent segment that share an endpoint not on the circle states that for the picture below secant segment QR and the tangent segment SR share and endpoint R, not on the circle. Then the length of SR squared is equal to the product of the lengths of QR and KR.

\[ x^2 = 16 \cdot 10 \]

So for the example above

\[ x^2 = 160 \]

\[ x = \sqrt{160} = 4\sqrt{10} \approx 12.6 \]

- How does the angle between a tangent to a circle and the line connecting the point of tangency and the center of the circle change as you move the tangent point?

**Instructional Strategies:** See G.C.1
Geometry: Circles (G-C)

Cluster: Understand and apply theorems about circles.

Standard: G.C.3 Construct the inscribed and circumscribed circles of a triangle, and prove properties of angles for a quadrilateral inscribed in a circle.

Suggested Standards for Mathematical Practice (MP):
MP.3 Construct viable arguments and critique the reasoning of others.
MP.5 Use appropriate tools strategically.

Connections:
Constructing inscribed and circumscribed circles of a triangle is an application of the formal constructions studied in G.CO.12.

Explanations and Examples: G.C.3

Define the terms inscribed, circumscribed, angle bisector, and perpendicular bisector.
Construct the inscribed circle whose center is the point of intersection of the angle bisectors (the incenter).
Construct the circumscribed circle whose center is the point of intersection of the perpendicular bisectors of each side of the triangle (the circumcenter).
Apply the Arc Addition Postulate to solve for missing arc measures.
Prove that opposite angles in an inscribed quadrilateral are supplementary.
Using definitions, properties, and theorems, prove properties of angles for a quadrilateral inscribed in a circle.
Students may use geometric simulation software to make geometric constructions.

Examples:
- The following diagram shows a circle that just touches the sides of a right triangle whose sides are 3 units, 4 units, and 5 units long.

![Diagram of a right triangle with a circle inscribed]

Explain why triangles AOX and Aoy are congruent.

a. What can you say about the measures of the line segments CX and CZ?
b. Find r, the radius of the circle. Explain your work clearly and show all your calculations.

Continued on next page
Explanations and Examples: G.C.3

- The following diagram shows a circle that just touches the sides of a right triangle whose sides are 5 units, 12 units, and 13 units long. Draw radius lines as in the previous task and find the radius of the circle in this 5, 12, 13 right triangle. Explain your work and show your calculations.

- Given the inscribed quadrilateral below prove that \( \angle B \) is supplementary to \( \angle D \).

- You have been asked to place a fire hydrant so that it is an equal distance from three locations indicated on the following map.

  a. Show how to fold your paper to physically construct this point as an intersection of two creases.
  b. Explain why the above construction works, and in particular why you only need to make two creases.

Solution: (This task connects to standards G.CO.12-13)

a. Fold and crease the paper so that line segment point \( A \) lands onto point \( B \). Do the same so that point \( A \) lands on point \( C \). The intersection of the two creases is the point we want.

b. Since the desired location is an equal distance from three non-collinear points, we are looking for the center of the circle passing through these three points. This corresponds to the center of the circle circumscribed about the triangle \( ABC \).

The center of the circumcircle, called the circumcenter, can be found by constructing the perpendicular bisectors of the three sides of the triangle (precisely the creases made in the paper on the previous step). Since the perpendicular bisectors are concurrent, it is sufficient to construct only two of the three perpendicular bisectors.

The concurrency of the perpendicular bisectors can be argued as follows: Let \( P \) be the green dot in the above diagram, the intersection of the perpendicular bisectors of \( AB \) and \( AC \). By virtue of \( P \) being on the perpendicular bisector of \( AB \), \( P \) is equidistant from \( A \) and \( B \), i.e., \( PA = PB \). Similarly, by virtue of being on the perpendicular bisector of \( AC \), we have \( PA = PC \). But this implies that \( PB = PC \), i.e., that \( P \) is also on the perpendicular bisector of \( BC \), demonstrating that \( P \) indeed lies on all three perpendicular bisectors.
Geometry: Circles \((G-C)\)

Cluster: Find arc lengths and areas of sectors of circles.

Standard: \(G.C.5\) Derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector.

Suggested Standards for Mathematical Practice (MP):
MP.2 Reason abstractly and quantitatively.   
MP.7 Look for and make use of structure. 
MP.3 Construct viable arguments and critique the reasoning of others. 
MP.6 Attend to precision.

Connections:
Formulas for area and circumference of a circle were developed in Grade 7. In this cluster the formulas are generalized to fractional parts of a circle and will prepare students for the study of trigonometry.

Explanations and Examples: \(G.C.5\)

Emphasize the similarity of circles. Note that by similarity of sectors with the same central angle, arc lengths are proportional to the radius. Use this as a basis for introducing radian as a unit of measure. It is not intended that it be applied to the development of circular trigonometry at this point.

Students can use geometric simulation software to explore angle and radian measure and derive the formula for the area of a sector.

Examples:

- Find the area of the sectors below. What general formula can you develop based on this information?
Explanations and Examples: G.C.5

- The amusement park has discovered that the brace that provides stability to the Ferris wheel has been damaged and needs work. The arc length of steel reinforcement that must be replaced is between the two seats shown below. If the sector area is 28.25 ft² and the radius is 12 feet, what is the length of steel that must be replaced? Describe the steps you used to find your answer.

- If the amusement park owners wanted to decorate each sector of this Ferris wheel with a different color of fabric, how much of each color fabric would they need to purchase? The area to be covered is described by an arc length of 5.9 feet. The circle has a radius of 15 feet. Describe the steps you used to find your answer.

Instructional Strategies: G.C.5

Begin by calculating lengths of arcs that are simple fractional parts of a circle (e.g. \( \frac{1}{6} \)), and do this for circles of various radii so that students discover a proportionality relationship.

Provide plenty of practice in assigning radian measure to angles that are simple fractional parts of a straight angle. Stress the definition of radian by considering a central angle whose intercepted arc has its length equal to the radius, making the constant of proportionality 1. Students who are having difficulty understanding radians may benefit from constructing cardboard sectors whose angles are one radian. Use a ruler and string to approximate such an angle.

Compute areas of sectors by first considering them as fractional parts of a circle. Then, using proportionality, derive a formula for their area in terms of radius and central angle. Do this for angles that are measured both in degrees and radians and note that the formula is much simpler when the angels are measured in radians.

Derive formulas that relate degrees and radians.

Introduce arc measures that are equal to the measures of the intercepted central angles in degrees or radians. Emphasize appropriate use of terms, such as, angle, arc, radian, degree, and sector.

Common Misconceptions: G.C.5

Sectors and segments are often used interchangeably in everyday conversation. Care should be taken to distinguish these two geometric concepts.

The formulas for converting radians to degrees and vice versa are easily confused. Knowing that the degree measure of given angle is always a number larger than the radian measure can help students use the correct unit.
Geometry: Expressing Geometric Properties with Equations (G-GPE)

Cluster: Translate between the geometric description and the equation for a conic section.

Standard: G.GPE.1 Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

Suggested Standards for Mathematical Practice (MP):
MP.2 Reason abstractly and quantitatively.
MP.3 Construct viable arguments and critique the reasoning of others.
MP.7 Look for and make use of structure. MP.8 Look for and express regularity in repeated reasoning.

Connections: G.GPE.1-2
In Grade 8 the Pythagorean theorem was applied to find the distance between two particular points. In high school, the application is generalized to obtain formulas related to conic sections. Quadratic functions and the method of completing the square are studied in the domain of interpreting functions. The methods are applied here to transform a quadratic equation representing a conic section into standard form.

Explanations and Examples: G.GPE.1
Connect G.GPE.1 to G.GPE.4 and reasoning with triangles, limited to right triangles, e.g., derive the equation for a line through two points using similar right triangles. Identify the center and radius of a circle given its equation. Draw a right triangle with a horizontal leg, a vertical leg, and the radius of a circle as its hypotenuse. Use the Pythagorean Theorem, the coordinates of a circle’s center, and the circle’s radius to write the equation of the circle. Convert an equation of a circle in general (quadratic) form to standard form by completing the square. Identify the center and radius of a circle given its equation. Students may use geometric simulation software to explore the connection between circles and the Pythagorean Theorem.

Examples:
- Write an equation for a circle with a radius of 2 units and center at (1, 3).
- Write an equation for a circle given that the endpoints of the diameter are (−2, 7) and (4, −8).
- Find the center and radius of the circle $4x^2 + 4y^2 − 4x + 2y − 1 = 0$.
- A circle is tangent to the x-axis and y-axis in the first quadrant. A point of tangency has coordinates (4, 0). Find the equation of the circle.
- A circle is inscribed in an equilateral triangle. The equilateral triangle lies in the first quadrant with one vertex at the origin and second vertex at $(4\sqrt{3}, 0)$.

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Instructional Strategies: G.GPE.1

Review the definition of a circle as a set of points whose distance from a fixed point is constant.

Review the algebraic method of completing the square and demonstrate it geometrically.

Illustrate conic sections geometrically as cross sections of a cone.

Use the Pythagorean theorem to derive the distance formula. Then, use the distance formula to derive the equation of a circle with a given center and radius, beginning with the case where the center is the origin.

Starting with any quadratic equation in two variables (x and y) in which the coefficients of the quadratic terms are equal, complete the squares in both x and y and obtain the equation of a circle in standard form.

Given two points, find the equation of the circle passing through one of the points and having the other as its center.

Import images of circle from fields from Google Earth into a coordinate grid system and find their equations.

Common Misconceptions: G.GPE.1-2

Because new vocabulary is being introduced in this cluster, remembering the names of the conic sections can be problematic for some students.

The Euclidean distance formula involves squared, subscripted variables whose differences are added. The notation and multiplicity of steps can be a serious stumbling block for some students.

The method of completing the square is a multi-step process that takes time to assimilate. A geometric demonstration of completing the square can be helpful in promoting conceptual understanding.
### Geometry: Expressing Geometric Properties with Equations (G-GPE)

#### Cluster: Translate between the geometric description and the equation for a conic section.

#### Standard: G.GPE.2

Derive the equation of a parabola given a focus and directrix.

#### Suggested Standards for Mathematical Practice (MP):

- MP.2 Reason abstractly and quantitatively.
- MP.3 Construct viable arguments and critique the reasoning of others.
- MP.7 Look for and make use of structure.
- MP.8 Look for and express regularity in repeated reasoning.

#### Connections:

See **G.GPE.1**  

#### Common Misconceptions:

See **G.GPE.1**

#### Explanations and Examples: G.GPE.2

The directrix should be parallel to a coordinate axis.

Find the distance from a point on the parabola \((x, y)\) to the directrix.

Find the distance from a point on the parabola \((x, y)\) to the focus using the distance formula (Pythagorean Theorem).

Equate the two distance expressions for a parabola to write its equation.

Identify the focus and directrix of a parabola when given its equation.

Students may use geometric simulation software to explore parabolas.

#### Examples:

- Write and graph an equation for a parabola with focus \((2, 3)\) and directrix \(y = 1\).

- Given the equation \(20(y - 5) = (x + 3)^2\), find the focus, vertex and directrix.

  _Solution:_ The vertex is at \((-3, 5)\) and to find the vertex we know that the constant of the unsquared term is 20. Since \(4p = 20\) then \(p = 5\). The focus is 5 units above the vertex at \((-3, 5+5)\) or \((-3, 10)\). The directrix is 5 units below the vertex so \(y = 0\).

- A parabola has focus \((-2, 1)\) and directrix \(y = -3\). Determine whether or not the point \((2, 1)\) is part of the parabola. Justify your answer.

#### Instructional Strategies: G.GPE.2

Define a parabola as a set of points satisfying the condition that their distance from a fixed point (focus) equals their distance from a fixed line (directrix). Start with a horizontal directrix and a focus on the \(y\)-axis, and use the distance formula to obtain an equation of the resulting parabola in terms of \(y\) and \(x^2\). Next use a vertical directrix and a focus on the \(x\)-axis to obtain an equation of a parabola in terms of \(x\) and \(y^2\). Make generalizations in which the focus may be any point, but the directrix is still either horizontal or vertical. Allow sufficient time for students to become familiar with new vocabulary and notation.

Given \(y\) as a quadratic equation of \(x\) (or \(x\) as a quadratic function of \(y\)), complete the square to obtain an equation of a parabola in standard form.

Identify the vertex of a parabola when its equation is in standard form and show that the vertex is halfway between the focus and directrix.

Investigate practical applications of parabolas and paraboloids.
Geometry: Expressing Geometric Properties with Equations (G-GPE)

Cluster: Use coordinates to prove simple geometric theorems algebraically.

Standard: G.GPE.4 Use coordinates to prove simple geometric theorems algebraically. For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point (1, \(\sqrt{3}\)) lies on the circle centered at the origin and containing the point (0, 2).

Suggested Standards for Mathematical Practice (MP):
MP.2 Reason abstractly and quantitatively.
MP.3 Construct viable arguments and critique the reasoning of others.
MP.7 Look for use and make of structure.

Connections: G.GPE.4-7
Rates of change and graphs of linear equations were studied in Grade 8 and generalized in the Functions and Geometry Conceptual Categories in high school. Therefore, an alternative way to define the slope of a line is to call it the tangent of an angle of inclination of the line.

Explanations and Examples: G.GPE.4
Represent the vertices of a figure in the coordinate plane using variables. Use coordinates to prove or disprove a claim about a figure. For example: use slope to determine if sides are parallel, intersecting, or perpendicular; use the distance formula to determine if sides are congruent or to decide if a point is inside a circle, outside a circle, or on the circle; use the midpoint formula or the distance formula to decide if a side has been bisected. Students may use geometric simulation software to model figures and prove simple geometric theorems.

Examples:
- Use slope and distance formula to verify the polygon formed by connecting the points (–3, –2), (5, 3), (9, 9), (1, 4) is a parallelogram.
- Prove or disprove that triangle \(ABC\) with coordinates \(A\) (–1, 2), \(B\) (1, 5), \(C\) (–2, 7) is an isosceles right triangle.
- Take a picture or find a picture which includes a polygon. Overlay the picture on a coordinate plane (manually or electronically). Determine the coordinates of the vertices. Classify the polygon. Use the coordinates to justify the classification.

Instructional Strategies: G.GPE.4-7
Review the concept of slope as the rate of change of the y-coordinate with respect to the x-coordinate for a point moving along a line, and derive the slope formula. Use similar triangles to show that every nonvertical line has a constant slope. Review the point-slope, slope-intercept and standard forms for equations of lines. Investigate pairs of lines that are known to be parallel or perpendicular to each other and discover that their slopes are either equal or have a product of –1, respectively.

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### Instructional Strategies: G.GPE.4-7

Pay special attention to the slope of a line and its applications in analyzing properties of lines.

Allow adequate time for students to become familiar with slopes and equations of lines and methods of computing them.

Use slopes and the Euclidean distance formula to solve problems about figures in the coordinate plane such as:

- Given three points, are they vertices of an isosceles, equilateral, or right triangle?
- Given four points, are they vertices of a parallelogram, a rectangle, a rhombus, or a square?
- Given the equation of a circle and a point, does the point lie outside, inside, or on the circle?
- Given the equation of a circle and a point on it, find an equation of the line tangent to the circle at that point.
- Given a line and a point not on it, find an equation of the line through the point that is parallel to the given line.
- Given a line and a point not on it, find an equation of the line through the point that is perpendicular to the given line.
- Given the equations of two non-parallel lines, find their point of intersection.

Given two points, use the distance formula to find the coordinates of the point halfway between them. Generalize this for two arbitrary points to derive the midpoint formula.

Use linear interpolation to generalize the midpoint formula and find the point that partitions a line segment in any specified ratio.

Given the vertices of a triangle or a parallelogram, find the equation of a line containing the altitude to a specified base and the point of intersection of the altitude and the base. Use the distance formula to find the length of that altitude and base, and then compute the area of the figure.

### Common Misconceptions: G.GPE.4-7

Students may claim that a vertical line has infinite slopes. This suggests that infinity is a number. Since applying the slope formula to a vertical line leads to division by zero, we say that the slope of a vertical line is undefined.

Also, the slope of a horizontal line is 0. Students often say that the slope of vertical and/or horizontal lines is “no slope,” which is incorrect.
# Geometry: Expressing Geometric Properties with Equations (G-GPE)

**Cluster:** Use coordinates to prove simple geometric theorems algebraically.

**Standard:** **G.GPE.5** Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point).

**Suggested Standards for Mathematical Practice (MP):**

- **MP.3** Construct viable arguments and critique the reasoning of others.
- **MP.8** Look for and express regularity in repeated reasoning.

**Connections:** See **G.GPE.4**

**Common Misconceptions:** See **G.GPE.4**

**Explanations and Examples:** **G.GPE.5**

Relate work on parallel lines to standard A.REI.5 involving systems of equations having no solution or infinitely many solutions.

Lines can be horizontal, vertical or neither.

Prove that the slopes of parallel lines are equal.

Prove that the product of the slopes of perpendicular lines is $-1$.

Write the equation of a line parallel or perpendicular to a given line, passing through a given point.

Students may use a variety of different methods to construct a parallel or perpendicular line to a given line and calculate the slopes to compare relationships.

**Examples:**

- Find the equation of a line perpendicular to $3x + 5y + 15$ through the point $(-3, 2)$.
- Find an equation of a line perpendicular to $y = 3x – 4$ that passes through $(3, 4)$.
- Verify that the distance between two parallel lines is constant. Justify your answer.

**Instructional Strategies:** See **G.GPE.4**

Allow students to explore and make conjectures about relationships between lines and segments using a variety of methods.

Discuss the role of algebra in providing a precise means of representing a visual image.
Cluster: *Use coordinates to prove simple geometric theorems algebraically.*

**Standard: G.GPE.6** Find the point on a directed line segment between two given points that partitions the segment in a given ratio.

**Suggested Standards for Mathematical Practice (MP):**
- MP.2 Reason abstractly and quantitatively.
- MP.7 Look for and make use of structure.
- MP.8 Look for and express regularity in repeated reasoning.

**Connections:** See G.GPE.4

**Common Misconceptions:** See G.GPE.4

**Explanations and Examples: G.GPE.6**
Students may use geometric simulation software to model figures or line segments.

**Examples:**

- Given $A(3, 2)$ and $B(6, 11)$,
  
  o Find the point that divides the line segment $AB$ two-thirds of the way from $A$ to $B$.

  *Solution:*
  The point two-thirds of the way from $A$ to $B$ has an $x$-coordinate two-thirds of the way from 3 to 6 and a $y$-coordinate two-thirds of the way from 2 to 11. So $(5, 8)$ is the point that is two-thirds from point $A$ to $B$.

  o Find the midpoint of the line segment $AB$.

  
  - For the line segment whose endpoints are $(0, 0)$ and $(4, 3)$, find the point that partitions the segment into a ratio of 3 to 2.

  *Solution:*
  
  \[
  x = \frac{(2 \cdot 0) + (3 \cdot 4)}{(3 + 2)} = \frac{12}{5}, \quad y = \frac{(2 \cdot 0) + (3 \cdot 3)}{(3 + 2)} = \frac{9}{5}, \text{ so the point is } \left( \frac{12}{5}, \frac{9}{5} \right)
  \]

**Instructional Strategies:** See G.GPE.4
Cluster: Use coordinates to prove simple geometric theorems algebraically.

Standard: G.GPE.7 Use coordinates to compute perimeters of polygons and areas of triangles and rectangles, e.g., using the distance formula. (★)

Suggested Standards for Mathematical Practice (MP):
MP.1 Make sense of problems and persevere in solving them.      MP.5 Use appropriate tools strategically.
MP.2 Reason abstractly and quantitatively.        MP.6 Attend to precision.

Connections: See G.GPE.4 Common Misconceptions: See G.GPE.4

Explanations and Examples: G.GPE.7
This standard provides practice with the distance formula and its connection with the Pythagorean Theorem. Use the coordinates of the vertices of a polygon graphed in the coordinate plane and use the distance formula to compute the perimeter. Use the coordinates of the vertices of triangles and rectangles graphed in the coordinate plane to compute the area. Students may use geometric simulation software to model figures.

Examples:
• Find the perimeter and area of a rectangle with vertices at C (−1, 1), D (3, 4), E (6, 0), F (2, −3). Round your answer to the nearest hundredth when necessary.

• Find the area and perimeter for the figure below.

• Calculate the area of triangle ABC with altitude \( \overline{CD} \), given \( A (−4, −2), B (8, 7), C (1, 8) \) and \( D (4, 4) \).
**Instructional Strategies:** See G.GPE.4

Graph polygons using coordinates. Explore perimeter and area of a variety of polygons, including convex, concave, and irregularly shaped polygons.

Given a triangle, use slopes to verify that the length and height are perpendicular. Find the area.

Find the area and perimeter of a real-world shape using a coordinate grid and Google Earth. Select a shape (yard, parking lot, school, etc.). Use the tool menu to overlay a coordinate grid. Use coordinates to find the perimeter and area of the shape selected. Determine the scale factor of the picture as related to the actual real-life view. Then find the actual perimeter and area.
Geometry: Geometric Measurement and Dimensions (G-GMD)

Cluster: Explain volume formulas and use them to solve problems.

Standard: G.GMD.1 Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone. Use dissection arguments, Cavalieri’s principle, and informal limit arguments.

Suggested Standards for Mathematical Practice (MP):
MP.3 Construct viable arguments and critique the reasoning of others. MP.4 Model with mathematics. MP.5 Use appropriate tools strategically.

Connections: G.GMD1; G.GMD.3
In Grade 8, students were required to know and use the formulas for volumes of cylinders, cones, and spheres. In Grade 7 students informally derived the formula for the area of a circle from the circumference. In this cluster those formulas are derived by a combination of concrete demonstrations and formal reasoning.

Explanations and Examples: G.GMD.1
Informal arguments for area and volume formulas can make use of the way in which area and volume scale under similarity transformations: when one figure in the plane results from another by applying similarity transformation with scale factor \(k\); its area is \(k^2\) times the area of the first. Similarly, volumes of solid figures scale \(k^3\) under a similarity transformation with scale factor \(k\).

Explain the formulas for the circumference of a circle and the area of circle by determining the meaning of each term or factor. Explain the formulas for the volume of a cylinder, pyramid and cone by determining the meaning of each term or factor.

Understand Cavalieri’s principle - if two solids have the same height and the same cross-sectional area at every level, then they have the same volume.

Examples:
• Use the diagram to give an informal argument for the formula for finding the area of a circle. (This concept was introduced in Grade 7).

• Justify using the given measurements to find the volume of the Great Pyramid of Giza. (Dimensions shown on the diagram below are in royal cubits).

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**Explanations and Examples: G.GMD.1**

- Explain why the volume of a cylinder is \( V = \pi r^2 h \).

- Prove that the right cylinder and the oblique cylinder have the same volume.

![Right cylinder and Oblique cylinder](image)

**Instructional Strategies: G.GMD.1 & G.GMD.3**

Revisit formulas \( C = \pi d \) and \( C = 2\pi r \). Observe that the circumference is a little more than three times the diameter of the circle. Briefly discuss the history of this number and attempts to compute its value.

Review alternative ways to derive the formula for the area of the circle \( A = \pi r^2 \). For example, cut a cardboard circular disk into 6 congruent sectors and rearrange the pieces to form a shape that looks like a parallelogram with two scalloped edges. Repeat the process with 12 sectors and note how the edges of the parallelogram look “straighter.” Discuss what would happen in the case as the number of sectors becomes infinitely large. Then calculate the area of a parallelogram with base \( \frac{1}{2} C \) and altitude \( r \) to derive the formula \( A = \pi r^2 \).

Wind a piece of string or rope to form a circular disk and cut it along a radial line. Stack the pieces to form a triangular shape with base \( C \) and altitude \( r \). Again discuss what would happen if the string became thinner and thinner so that the number of pieces in the stack became infinitely large. Then calculate the area of the triangle to derive the formula \( A = \pi r^2 \).

Introduce Cavalieri’s principle using a concrete model, such as a deck of cards. Use Cavalieri’s principle with cross sections of cylinders, pyramids, and cones to justify their volume formulas.

For pyramids and cones, the factor \( \frac{1}{3} \) will need some explanation. An informal demonstration can be done using a volume relationship set of plastic shapes that permit one to pour liquid or sand from one shape into another. Another way to do this for pyramids is with Geoblocks. The set includes three pyramids with equal bases and altitudes that will stack to form a cube. An algebraic approach involves the formula for the sum of squares \((1^2 + 2^2 + \ldots + n^2)\).

After the coefficient \( \frac{1}{3} \) has been justified for the formula of the volume of the pyramid \( (A = \frac{1}{3}Bh) \), one can argue that it must also apply to the formula of the volume of the cone by considering a cone to be a pyramid that has a base with infinitely many sides.

The formulas for volumes of cylinders, pyramids, cones and spheres can be applied to a wide variety of problems such as finding the capacity of a pipeline; comparing the amount of food in cans of various shapes; comparing capacities of cylindrical, conical and spherical storage tanks; using pyramids and cones in architecture; etc. Use a combination of concrete models and formal reasoning to develop conceptual understanding of the volume formulas.

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Common Misconceptions: G.GMD.1 & G.GMD.3

An informal survey of students from elementary school through college showed the number pi to be the mathematical idea about which more students were curious than any other. There are at least three facets to this curiosity: the symbol π itself, the number 3.14159..., and the formula for the area of a circle. All of these facets can be addressed here, at least briefly.

Many students want to think of infinity as a number. Avoid this by talking about a quantity that becomes larger and larger with no upper bound.

The inclusion of the coefficient 1/3 in the formulas for the volume of a pyramid or cone and 4/3 in the formula for the volume of a sphere remains a mystery for many students. In high school, students should attain a conceptual understanding of where these coefficients come from. Concrete demonstrations, such as pouring water from one shape into another should be followed by more formal reasoning.
Cluster:  *Explain volume formulas and use them to solve problems.*

**Standard:**  **G.GMD.3**  Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems. (★)

**Suggested Standards for Mathematical Practice (MP):**
MP.1  Make sense of problems and persevere in solving them.          MP.4  Model with mathematics.
MP.2  Reason abstractly and quantitatively.                     MP.5  Use appropriate tools strategically.
MP.3  Construct viable arguments and critique the reasoning of others.

**Connections:**  See **G.GMD.1**          **Common Misconceptions:**  See **G.GMD.1**

**Explanations and Examples:  ** **G.GMD.3**

Missing measures can include but are not limited to slant height, altitude, height, diagonal of a prism, edge length, and radius.
Find the volume of cylinders, pyramids, cones and spheres in contextual problems.

**Examples:**

- Determine the volume of the figure below.

![Diagram of a cone with dimensions 40 cm, 4 cm.]

- Find the volume of a cylindrical oatmeal box.

- Given a three-dimensional object, compute the effect on volume of doubling or tripling one or more dimension(s). For example, how is the volume of a cone affected by doubling the height?

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Explanations and Examples: G.GMD.3

- Janine is planning on creating a water-based centerpiece for each of the 30 tables at her wedding reception. She has already purchased a cylindrical vase for each table. The radius of the vases is 6 cm. and the height is 28 cm. She intends to fill them half way with water and then add a variety of colored marbles until the waterline is approximately three-quarters of the way up the cylinder. She can buy bags of 100 marbles in 2 different sizes, with radii of 9mm or 12 mm. A bag of 9 mm marbles costs $3, and a bag of 12 mm marbles costs $4.

a. If Janine only bought 9 mm marbles how much would she spend on marbles for the whole reception?

   What if Janine only bought 12 mm marbles? (Note: 1 cm$^3$ = 1 mL)

b. Janine wants to spend at most d dollars on marbles. Write a system of equalities and/or inequalities that she can use to determine how many marbles of each type she can buy.

c. Based on your answer to part b. How many bags of each size marble should Janine buy if she has $180 and wants to buy as many small marbles as possible?

Solution:

a. We are looking to fill one fourth of the cylinder's volume with marbles, a volume given by

$$\frac{1}{3}\pi r^2h = \frac{1}{3}\pi \cdot 6^2 \cdot 28 = 252\pi \text{ cm}^3.$$ 

Using the formula for the volume of the sphere, each of the 9mm (=0.9cm) marbles has a volume of $\frac{4}{3}\pi (0.9)^3$ or 0.972$\pi \text{ cm}^3$. So to fill the desired volume requires $\frac{252\pi \text{ cm}^3}{0.972\pi \text{ cm}^3} = 260$ marbles. To obtain 260 marbles per table for each of the 30 tables, Janine needs to purchase 260 $\cdot$ 30 = 7800 marbles. At 100 marbles per $3$-bag, this requires 78 bags of marbles, for a total cost of $3 \cdot 78 = 234$ dollars. We repeat for the 12mm marbles similarly. Each of the 12mm marbles has a volume of $\frac{4}{3}\pi (1.2)^3$ or 2.304$\pi \text{ cm}^3$. To fill that volume requires $\frac{252\pi \text{ cm}^3}{2.304\pi \text{ cm}^3} = 110$ marbles. Thus 30 tables require 3300 marbles, which in turn requires 33 bags of 12mm-marbles. At a cost of $4$ per bag, we arrive at a total cost of $132.

b. We have two constraints: 1) that we don't spend more than d dollars, and 2) that we acquire enough volume of marbles to fill the cylinders to their desired level (approximately a quarter of the cylinder, as in part a). Let $s$ be the number of bags of smaller (9mm) marbles and $b$ be the number bags of bigger (12mm) marbles.

The first constraint corresponds to the inequality

$$3 \cdot s + 4 \cdot b \leq d,$$

and the second (using calculations from part (a)) is the approximate equality

$$100 \cdot s \cdot 0.972\pi + 100 \cdot b \cdot 2.304\pi \approx 252\pi \cdot 30.$$

which we can re-arrange more simply as

$$97.2 \cdot s + 230.4 \cdot b = 7560.$$ 

c. Since we learned in part (a) that it would require $234$ to do the reception entirely with small marbles, she will certainly have to spend all $180 when maximizing the number of small marbles. Thus we can precede by solving the system of linear equations below (we have replaced approximations with equalities for the sake of solving the system, and then consider rounding below.)

$$3 \cdot s + 4 \cdot b = 180$$

$$97.2 \cdot s + 230.4 \cdot b = 7560.$$ 

Multiplying the top row by $\frac{97.2}{3} = 32.4$, we get

$$-97.2 \cdot s - 129.6 \cdot b = -5832$$

$$97.2 \cdot s + 230.4 \cdot b = 7560.$$
Cluster: Visualize relationships between two-dimensional and three-dimensional objects.

Standard: G.GMD.4 Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.

Suggested Standards for Mathematical Practice (MP):
MP.4 Model with mathematics.
MP.5 Use appropriate tools strategically.
MP.7 Look for and make use of structure.

Connections:
Slices of rectangular prisms and pyramids were explored in Grade 7. In high school, the concept is extended to a wider class of solids.
Students who eventually take calculus will learn how to compute volumes of solids of revolution by a method involving cross-sectional disks.

Explanations and Examples: G.GMD.4
Given a three-dimensional object, identify the shape made when the object is cut into cross-sections.
When rotating a two-dimensional figure, such as a square, know the three-dimensional figure that is generated, such as a cylinder. Understand that a cross section of a solid is an intersection of a plane (two-dimensional and a solid (three-dimensional)).
Students may use geometric simulation software to model figures and create cross sectional views.

Examples:
- Identify the shape of the vertical, horizontal, and other cross sections of a rectangular prism.

• Identify the shape of the vertical, horizontal, and other cross sections of a rectangular prism.

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Explanations and Examples: G.GMD.4

- The official diameter of a tennis ball, as defined by the International Tennis Federation, is at least 2.575 inches and at most 2.700 inches. Tennis balls are sold in cylindrical containers that contain three balls each. To model the container and the balls in it, we will assume that the balls are 2.7 inches in diameter and that the container is a cylinder the interior of which measures 2.7 inches in diameter and $3 \times 2.7 = 8.1$ inches high.

a. Lying on its side, the container passes through an X-ray scanner in an airport. If the material of the container is opaque to X-rays, what outline will appear? With what dimensions?

b. If the material of the container is partially opaque to X-rays and the material of the balls is completely opaque to X-rays, what will the outline look like (still assuming the can is lying on its side)?

c. The *central axis* of the container is a line that passes through the centers of the top and bottom. If one cuts the container and balls by a plane passing through the central axis, what does the intersection of the plane with the container and balls look like? (The intersection is also called a *cross section*. Imagine putting the cut surface on an ink pad and then stamping a piece of paper. The stamped image is a picture of the intersection.)

d. If the can is cut by a plane parallel to the central axis, but at a distance of 1 inch from the axis, what will the intersection of this plane with the container and balls look like?

e. If the can is cut by a plane parallel to one end of the can—a horizontal plane—what are the possible appearances of the intersections?

f. A cross-section by a horizontal plane at a height of $1.35 + w$ inches from the bottom is made, with $0 < w < 1.35$ (so the bottom ball is cut). What is the area of the portion of the cross section inside the container but outside the tennis ball?

g. Suppose the can is cut by a plane parallel to the central axis but at a distance of $w$ inches from the axis ($0 < w < 1.35$). What fractional part of the cross section of the container is inside of a tennis ball?

**Solution:** (This task connects with domain G.MG)

a. The shadow is a rectangle measuring 2.7 inches by 8.1 inches.

b. The shadow is a light rectangle ($2.7 \times 8.1$ inches) with three disks inside. It looks like a traffic light:

c. The image is similar to the previous one, but now only the outlines are seen:

*Continued on next page*
d. The intersection with the container is a narrower rectangle. The intersections with the balls are smaller circles. Because each ball touches the container along its whole “equator,” the circles must touch the long sides of the rectangle:

\[ \text{\includegraphics[width=0.5\textwidth]{example_d}} \]

Because each ball touches the container along its whole “equator,” the circles must touch the long sides of the rectangle:

\[ \text{\includegraphics[width=0.5\textwidth]{example_e}} \]

e. The intersections are two concentric circles, except when \( w = 0, 2, 7, 5.4, 8.1 \) and when \( w = 1.35, 4.05, 6.75 \). In the former case, we see a circle (from the container) and a point (where the plane touches a sphere). In the latter case, we see a single circle corresponding to a place where the equator of a ball touches the container.

\[ \text{\includegraphics[width=0.5\textwidth]{example_f}} \]

f. The intersection of the plane with the interior of the container is a disk of radius 1.35 inches. Its area is \( \pi(1.35)^2 \text{in}^2 \). The intersection with the ball is a smaller disk that is contained in the first disk. The radius \( r \) of the smaller disk is the square root of \( (1.35)^2 - w^2 \), as we see from the diagram below depicting the intersection of a plane through the central axis of the container with the bottom ball. Thus, the area of the smaller disk is \( \pi((1.35)^2 - w^2) \). Accordingly, the area inside the larger disk but outside the smaller is \( \pi w^2 \), provided that \( 0 \leq w \leq 1.35 \). (It is notable that the radius of the ball does not appear explicitly in the expression for this annular area.)

\[ \text{\includegraphics[width=0.5\textwidth]{example_g}} \]

g. Referring to Problem d), we see that we wish to find the ratio of the total area of three congruent disks to the area of a rectangle, one of whose dimensions is equal to the diameter of the disks. The same picture used in the previous problem, but interpreted as a view from one end of the container, gives us the radius of the small disks — namely, \( \sqrt{(1.35)^2 - w^2} \), so the total area of the disks is \( 3\pi((1.35)^2 - w^2) \).

The area of the rectangle is \( (8.1)2\sqrt{(1.35)^2 - w^2} \). So, the ratio is

\[ \frac{3\pi((1.35)^2 - w^2)}{(8.1)2\sqrt{(1.35)^2 - w^2}} = \frac{\pi\sqrt{(1.35)^2 - w^2}}{5.4} \]

Continued on next page

195
**Instructional Strategies:** G.GMD.4

Review vocabulary for names of solids (e.g., right prism, cylinder, cone, sphere, etc.).

Slice various solids to illustrate their cross sections. For example, cross sections of a cube can be triangles, quadrilaterals or hexagons. Rubber bands may also be stretched around a solid to show a cross section.

Cut a half-inch slit in the end of a drinking straw, and insert a cardboard cutout shape. Rotate the straw and observe the three-dimensional solid of revolution generated by the two-dimensional cutout.

Java applets on some web sites can also be used to illustrate cross sections or solids of revolution.

Encourage students to create three-dimensional models to be sliced and cardboard cutouts to be rotated. Students can also make three-dimensional models out of modeling clay and slice through them with a plastic knife.

**Common Misconceptions:** G.GMD.4

Some cross sections are more difficult to visualize than others. For example, it is often easier to visualize a rectangular cross section of a cube than a hexagonal cross section.

Generating solids of revolution involves motion and is difficult to visualize by merely looking at drawings.
Cluster: Apply geometric concepts in modeling situations.

Standard: G.MG.1  Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder). (★)

Suggested Standards for Mathematical Practice (MP):
MP.4  Model with mathematics.                        MP.7  Look for and make use of structure.
MP.5  Use appropriate tools strategically.

Connections: G.MG.1-3
Modeling activities are a good way to show connections among various branches of mathematics.

Explanations and Examples: G.MG.1
Focus on situations that require relating two- and three- dimensional objects. Estimate measures (circumference, area, perimeter, volume) of real-world objects using comparable geometric shapes or three-dimensional objects. Apply the properties of geometric figures to comparable real-world objects (e.g., The spokes of a wheel of a bicycle are equal lengths because they represent the radii of a circle). Students may use simulation software and modeling software to explore which model best describes a set of data or situation.

Examples:
- How can you model objects in your classroom as geometric shapes?

- Picture a roll of toilet paper; assume that the paper in the roll is very tightly rolled. Assuming that the paper in the roll is very thin, find a relationship between the thickness of the paper, the inner and outer radii of the roll, and the length of the paper in the roll.
  Express your answer as an algebraic formula involving the four listed variables.

The purpose of this task is to engage students in geometric modeling, and in particular to deduce algebraic relationships between variables stemming from geometric constraints. The modeling process is a challenging one, and will likely elicit a variety of attempts from the students. Teachers should expect to spend time guiding students away from overly complicated models. Similarly, the task presents one solution, but alternatives abound: For example, students could imagine slicing the roll along a radius, unraveling the cross-section into a sequence of trapezoids whose area can be computed.

Continued on next page
Explanations and Examples: G.MG.1

Solution:

We begin by labeling the variables, for which the above diagrams may be useful. Let \( t \) denote the thickness of the paper, let \( r \) denote the inner radius, let \( R \) denote the outer radius and let \( L \) denote the length of the paper, all measured in inches. We now consider the area \( A \), measured in square inches, of the annular cross-section displayed at the top of the first image, consisting of concentric circles. Namely, we see that this area can be expressed in two ways: First, since this area is the area of the circle of radius \( R \) minus the area of the circle of radius \( r \), we learn that

\[
A = \pi (R^2 - r^2).
\]

Second, if the paper were unrolled, laid on a (very long) table and viewed from the side, we would see a very long thin rectangle. When the paper is rolled up, this rectangle is distorted, but -- assuming \( r \) is large in comparison to \( t \) -- the area of the distorted rectangle is nearly identical to that of the flat one. As in the second figure, the formula for the area of a rectangle now gives

\[
A = t \cdot L.
\]

Comparing the two formulas for \( A \), we find that the four variables are related by:

\[
t \cdot L = \pi (R^2 - r^2).
\]

Instructional Strategies: G.MG.1-3

Genuine mathematical modeling typically involves more than one conceptual category. For example, modeling a herd of wild animals may involve geometry, measurement, proportional reasoning, estimation, probability and statistics, functions, and algebra. It would be somewhat misleading to try to teach a unit with the title of “modeling with geometry.” Instead, these standards can be woven into other content clusters.

A challenge for teaching modeling is finding problems that are interesting and relevant to high school students and, at the same time, solvable with the mathematical tools at the students’ disposal. The resources listed below are a beginning for addressing this difficulty.

Common Misconceptions: G.MG.1-3

When students ask to see “useful” mathematics, what they often mean is, “Show me how to use this mathematical concept or skill to solve the homework problems.” Mathematical modeling, on the other hand, involves solving problems in which the path to the solution is not obvious. Geometry may be one of several tools that can be used.
## Geometry: Modeling with Geometry (G-MG)

### Cluster: Apply geometric concepts in modeling situations. (★)

### Standard: G.MG.2
Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot). (★)

### Suggested Standards for Mathematical Practice (MP):

- MP.1 Make sense of problems and persevere in solving them.
- MP.4 Model with mathematics.
- MP.5 Use appropriate tools strategically.

### Connections: See G.MG.1

### Common Misconceptions: See G.MG.1

### Explanations and Examples: G.MG.1

Decide whether it is best to calculate or estimate the area or volume of a geometric figure and perform the calculation or estimation. Students may use simulation software and modeling software to explore which model best describes a set of data or situation.

### Examples:

- Wichita, Kansas has 344,234 people within 165.9 square miles. What is Wichita’s population density?

- Consider the two boxes below. Each box has the same volume. If each ball has the same mass, which box would weight more? Why

- A King Size waterbed has the following dimensions 72 in. X 84 in. X 9.5in. It takes 240.7 gallons of water to fill it which would weigh 2071 pounds. What is the weight of a cubic foot of water?

### Instructional Strategies: See G.MG.1
Geometry: Modeling with Geometry (G-MG)

Cluster: Apply geometric concepts in modeling situations.

Standard: **G.MG.3**  Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios). (★)

Suggested Standards for Mathematical Practice (MP):
MP.1 Make sense of problems and persevere in solving them. MP.4 Model with mathematics.
MP.5 Use appropriate tools strategically.

Connections: See **G.MG.1**

Common Misconceptions: See **G.MG.1**

Explanations and Examples: **G.MG.3**
Create a visual representation of a design problem and solve using a geometric model (graph, equation, table, formula).
Interpret the results and make conclusions based on the geometric model.
Students may use simulation software and modeling software to explore which model best describes a set of data or situation.

Examples:
- Given one geometric solid, design a different geometric solid that will hold the same amount of substance (e.g., a cone to a prism).
- This paper clip is just over 4 cm long.

How many paper clips like this may be made from a straight piece of wire 10 meters long?

In this task, a typographic grid system serves as the background for a standard paper clip. A metric measurement scale is drawn across the bottom of the grid and the paper clip extends in both directions slightly beyond the grid. Students are given the approximate length of the paper clip and determine the number of like paper clips made from a given length of wire. Extending the paper clip beyond the grid provides an opportunity to include an estimation component in the problem. In the interest of open-ended problem solving, no scaffolding or additional questions are posed in this task. The paper clip modeled in this problem is an actual large standard paper clip.
Explanations and Examples: G.MG.3

Sample Response:

One approach is to divide the paper clip into vertical regions, and then to use the measurement grid to determine the length of the straight sections and estimate the length of the curved sections using a string or thin wire in conjunction with the measurement scale provided. One such division is accomplished using three vertical dividers splitting the paper clip into four distinct regions as shown.

<table>
<thead>
<tr>
<th>Regions</th>
<th>Number of Linear Sections</th>
<th>Number of Curved Sections</th>
</tr>
</thead>
<tbody>
<tr>
<td>Region A</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>Region B</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>Region C</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Region D</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

The lengths of the linear sections were determined using the gridlines. The estimations of the lengths of the curved sections were determined using a string or thin wire in conjunction with the measurement scale provided.

<table>
<thead>
<tr>
<th>Regions</th>
<th>Measurement of Linear Sections (listed from top to bottom)</th>
<th>Estimated Measurement of Curved Sections</th>
</tr>
</thead>
<tbody>
<tr>
<td>Region A</td>
<td></td>
<td>2.5cm</td>
</tr>
<tr>
<td>Region B</td>
<td>1.8cm, 1.5cm, 1.8cm, 1.5cm</td>
<td>2.5cm</td>
</tr>
<tr>
<td>Region C</td>
<td>0.8cm, 0.8cm</td>
<td>1.6cm</td>
</tr>
<tr>
<td>Region D</td>
<td></td>
<td>2.5cm</td>
</tr>
</tbody>
</table>

The length of wire needed to manufacture one paper clip is now approximately:

\[1.8 \text{ cm} + 1.5 \text{ cm} + 1.8 \text{ cm} + 1.5 \text{ cm} + 0.8 \text{ cm} + 0.8 \text{ cm} + 2.5 \text{ cm} + 1.6 \text{ cm} + 2.5 \text{ cm} = 14.8 \text{ cm}\]

The length of the straight piece of wire is 10 meters. Since 1 meter is the same as 100 centimeters, 10 meters is \(10 \cdot 100 = 1000\) centimeters. Finally, we find that at 14.8 cm per paper clip, 1000 centimeters will produce approximately

\[\frac{1000}{14.8} \approx 67.6\] paper clips.

Since we can only make a whole number of paper clips, we conclude that approximately 67 paper clips may be manufactured from a straight piece of wire 10 meters in length.

Instructional Strategies: See G.MG.1